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S-duality, deconstruction and confinement for a marginal deformation of $\mathcal{N} = 4$ SUSY Yang-Mills

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Abstract: We study an exactly marginal deformation of $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group $\mathrm{U}(N)$ using field theory and string theory methods. The classical theory has a Higgs branch for rational values of the deformation parameter. We argue that the quantum theory also has an S-dual confining branch which cannot be seen classically. The low-energy effective theory on these branches is a six-dimensional non-commutative gauge theory with sixteen supercharges. Confinement of magnetic and electric charges, on the Higgs and confining branches respectively, occurs due to the formation of BPS-saturated strings in the low energy theory. The results also suggest a new way of deconstructing Little String Theory as a large-N limit of a confining gauge theory in four dimensions.

Keywords: Field Theories in Higher Dimensions, 1/N Expansion, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Recent years have seen rapid progress in our understanding of supersymmetric (SUSY) gauge dynamics. In four dimensions, the best understood model is $\mathcal{N}=4$ SUSY Yang-Mills theory with gauge group SU(N) which exhibits exact electric-magnetic duality and has a dual large-N formulation as IIB string theory on $AdS_5 \times S^5$. On the other hand, the $\mathcal{N}=4$ theory only has conformal and Coulomb phases, and thus has little in common with non-supersymmetric Yang-Mills theory in its confining phase. In this paper (and a subsequent one [1]) we will study a system which is as close as possible to the $\mathcal{N}=4$

theory and shares some of its special properties. Specifically, we will deform the $\mathcal{N}=4$ lagrangian by adding an exactly marginal operator which preserves $\mathcal{N}=1$ supersymmetry. The resulting theory also exhibits many new features, including the emergence of extra dimensions and a confining phase with BPS-saturated strings. There are also interesting limits where the confining phase has a dual description in terms of six-dimensional Little String Theory [51]. We discuss similarities and differences between the marginally-deformed theory and the better understood case of a relevant deformation (the $\mathcal{N}=1^*$ theory). In particular, several phenomena which first emerged in Polchinski and Strassler's analysis of the $\mathcal{N}=1^*$ theory [2] will appear again here in a new and more tractable setting.

In terms of $\mathcal{N}=1$ multiplets, the theory we will consider contains a single vector multiplet V and three chiral multiplets Φ_i $i=1,2,3$ in the adjoint representation of the gauge group. The classical superpotential is,

$$\mathcal{W}=i\kappa\,\mathrm{Tr}_N\left[e^{i\frac{\beta}{2}}\Phi_1\Phi_2\Phi_3-e^{-i\frac{\beta}{2}}\Phi_1\Phi_3\Phi_2\right]. \tag{1.1}$$

The $\mathcal{N}=4$ theory is recovered by setting $\beta=0$ and $\kappa=1$. As we review in section 2, there is known to be a surface of renormalization group fixed points in coupling-constant space which includes the $\mathcal{N}=4$ point [3]. We will refer to the corresponding family of superconformal field theories collectively as the β-deformed theory. For earlier discussions of the finiteness of this model see [4] and for other relevant work see [5]–[15]. In this paper, we will mainly be interested in the β-deformed theory with gauge group $\mathrm{U}(N)$. However we also discuss the $\mathrm{SU}(N)$ theory, highlighting some important differences between the two cases.

Both theories already exhibit some interesting features at the classical level. In particular, new Higgs branches appear when the deformation parameter $\beta/2\pi$ takes any real rational value (for appropriate values of N). We will focus on a particular Higgs branch of the $\mathrm{U}(N)$ theory, denoted $\mathcal{H}_m$ which occurs when $\beta=2\pi/n$ with $N=mn$. At a generic point on this branch, the $\mathrm{U}(N)$ gauge group is broken to $\mathrm{U}(m)$ at a scale[1] v set by the scalar vacuum expectation value (VEV). A surprising aspect of the classical physics on this branch is the appearance of additional spacetime dimensions at low energy. In section 4, we calculate the exact classical spectrum of the theory and show that it has a natural Kaluza-Klein interpretation in terms of a six-dimensional theory. Using the observations of [10], we can rewrite the action of the four-dimensional $\mathrm{U}(N)$ theory as that of a six-dimensional $\mathrm{U}(m)$ theory with two spatial dimensions discretized on an $n\times n$ lattice with periodic boundary conditions. More precisely, the theory is a lattice regularization of a six-dimensional non-commutative $\mathrm{U}(m)$ gauge theory with $\mathcal{N}=(1,1)$ supersymmetry. The theory has lattice spacing $\varepsilon\sim\mathrm{v}^{-1}$ and is compactified on a two-dimensional torus of radii $R\sim n\varepsilon$. On length-scales much larger than ε, the classical lattice action is well approximated by its continuum counterpart which is a six-dimensional $\mathcal{N}=(1,1)$ theory with gauge group $\mathrm{U}(m)$ compactified on the non-commutative torus, T^2_Θ. The dimensionless non-commutativity parameter Θ takes the value $1/n$.

[1]In fact, the VEVs of the three complex scalars are independent on this branch and can provide more than one relevant scale. We discuss this more general case in sections 3 and 4 below.

This kind of equivalence between a lattice gauge theory and a lower-dimensional theory in its Higgs phase is known as deconstruction [16, 17] (see also [18, 19]). The relation to the original version of deconstruction, which yields an ordinary commutative lattice gauge theory, was explained in [10]. The appearance of extra supersymmetry at low-energies and at large-N is a familar feature of deconstruction. In the present context it has a very prosaic explanation: the theory on $\mathcal{H}_m$ has deformation parameter $\beta = 2\pi/n = 2\pi m/N$ which goes to zero as $N \to \infty$ and we recover the $\mathcal{N} = 4$ lagrangian with sixteen supercharges for any fixed value of the gauge coupling. Note that we are expanding around a background which is not a vacuum of the $\mathcal{N} = 4$ theory for any finite values N, but becomes so in the limit $N \to \infty$. The enhancement of supersymmetry allows us to argue that the classical spectrum of Kaluza-Klein states becomes exact in the $N \to \infty$ limit.[2]

The appearance of a non-commutative theory in six dimensions is also very reminiscent of the $\mathcal{N} = 1^*$ theory. The latter theory also has a vacuum in which $\mathrm{U}(N)$ is Higgsed down to $\mathrm{U}(m)$, with $m|N$, although the resulting physics is quite different because the unbroken $\mathrm{U}(m)$ is confined at low energies. The string theory realization of this ground state given in [2] involves N D3 branes polarized [47] by external Ramond-Ramond fields into m D5 branes wrapped on a two-sphere (see also [48]). The D3 brane charge appears as a background B_{NS} two-form potential leading to a six-dimensional non-commutative $\mathrm{U}(m)$ gauge theory on the D5 world-volume. As discussed in section 8 below, the Higgs branch $\mathcal{H}_m$ of the β-deformed theory has a very similar string theory realization in which the m D5-branes are wrapped on a two-dimensional torus rather than a sphere. The different physics of the two cases can be traced to this distinction. Toroidal compactification with periodic boundary conditions preserves the full sixteen supercharges of the D5 brane theory (although the lattice regularization implied by finite N does not). At energies far below the compactification scale, the $\mathrm{U}(m)$ theory on the Higgs branch therefore reduces to a conformal $\mathcal{N} = 4$ super Yang-Mills in four dimensions. In contrast, spherical compactification breaks most of the supersymmetry, leaving an $\mathcal{N} = 1$ theory in four dimensions which becomes strongly-coupled and confining in the IR (for $m > 1$).

By choosing the UV coupling to be small we can ensure that the theory on the Higgs branch $\mathcal{H}_m$ is weakly-coupled for all energy scales. In this regime, the low-energy effective theory is the six-dimensional non-commutative gauge theory described above which is also weakly-coupled throughout its range of validity justifying a semiclassical analysis. In particular, one can find an explicit map between the fields of the original four-dimensional gauge theory and those of the six-dimensional effective theory. As we explain in section 5, this map is a consequence of the standard Morita equivalence of gauge theory on the non-commutative torus T^2_Θ.

The low-energy effective theory has classical solutions corresponding to BPS-saturated strings. These solutions are obtained by embedding four-dimensional non-commutative $\mathrm{U}(m)$ Yang-Mills instantons as static solutions in six dimensions. The resulting strings have tension $T \sim \mathrm{v}^2/g^2 n$. In section 5, we use the Morita equivalence of the low-energy theory on T^2_Θ to show that these solutions correspond to magnetic flux tubes in the original

[2] Throughout this paper the $N \to \infty$ limit refers to the limit $n \to \infty$ with m fixed and $N = mn$.

JHEP08(2004)043

U(N) gauge theory. As usual, the formation of finite-tension magnetic flux-tubes indicates the confinement of external magnetic charges. At large-N, the core-size of the flux tube is always much larger than the lattice spacing justifying the use of the continuum low-energy theory. Using the Morita map, standard facts about non-commutative instantons translate into interesting predictions for the flux tubes of this theory. On the Higgs branch $\mathcal{H}_m$, where the unbroken gauge group is U(m) the flux tubes have a variable core size corresponding to the scale-size of the instanton. Increasing the core-size allows the magnetic flux lines of the unbroken U(m) to spread out. There are also interesting vacua where the gauge group is further broken down to $\mathrm{U}(1)^m$. In these vacua we find BPS magnetic flux tubes of fixed core-size.[3]

Another remarkable feature of the β-deformed theory is that it has an exact $\mathrm{SL}(2,\mathbb{Z})$ duality inherited from the S-duality of the underlying $\mathcal{N}=4$ theory. The evidence for this duality is reviewed in section 2 below. This includes a linearized analysis of the perturbation and also the exact effective superpotential for a massive version of the β-deformed theory derived in [15]. The $\mathrm{SL}(2,\mathbb{Z})$ generator corresponding to electric-magnetic duality relates the theory with coupling τ and deformation parameter $\beta = 2\pi/n$ to a dual theory with coupling $\tilde{\tau} = -1/\tau$ and deformation parameter[4] $\tilde{\beta} = 2\pi\tilde{\tau}/n$. The electric Higgs branch $\mathcal{H}_m$ in the former theory is mapped to a dual magnetic Higgs branch $\tilde{\mathcal{H}}_m$ in the latter. The magnetic branch occurs at a complex value of β and does not coincide with any vacuum states of the classical theory. The other elements of $\mathrm{SL}(2,\mathbb{Z})$ map the Higgs branch onto an infinite family of new branches realised in various oblique confining phases. The existence of these new branches can be demonstrated directly by taking an appropriate massless limit of the results in [15].

The S-duality described above maps the weak-coupling limit of the electric Higgs branch theory to a dual regime of the magnetic Higgs branch theory. In the dual picture, the classical BPS strings of the six-dimensional effective theory correspond to electric flux-tubes which confine electric charges. The confinement is partial in the sense that it only applies objects which are charged under the broken generators of the magnetic gauge group. This corresponds to a phase where the original electric gauge group U(N) is confined down to a U(m) subgroup. We also discuss phases where U(N) is further confined down to $\mathrm{U}(1)^m$ which are S-dual to the Higgs phases with unbroken gauge group $\mathrm{U}(1)^m$. The confining phase of the β-deformed theory is novel for several reasons. Apart from the appearance of extra dimensions and $\mathcal{N}=4$ supersymmetry at low energy, confinement occurs in a phase where conformal invariance is spontaneously broken leading to massless Goldstone bosons.[5] Importantly, as the theory contains only adjoint fields, the Higgs and confining phases are genuinely distinct.

In the semiclassical regime discussed above long strings appear as classical solitons in the low-energy theory. Thus the mass scale set by the string tension is large compared

[3] For other recent work on non-abelian flux tubes in deformations of $\mathcal{N}=4$ SUSY Yang-Mills see [20].

[4] This form of the transformation is only correct for $n \gg 1$. More generally τ recieves a finite algebraic renormalisation which is given in (2.12) below.

[5] Of course the coexistence of confinement and massless Goldstone modes is a well-known feature of QCD with massless quarks and is not, by itself, surprising.

with the masses of elementary quanta. An extrapolation of the semiclassical formula for the tension indicates that there should also be other regimes where the confining strings become light. The fact that the strings become BPS saturated with respect to the enlarged supersymmetry as $N \to \infty$ suggests that this extrapolation should be reliable for large-N. In the phase discussed above where the gauge group is confined to $\mathrm{U}(1)^m$ and we have flux tubes of fixed core-size, one might expect an infinite tower of states corresponding to the excitations of the light string. This would match the expected behaviour of a large-N confining gauge theory with adjoint fields, where the spectrum should include an infinite tower of glueball states. In section 9 below, we will identify a regime of parameters where we believe that this behaviour can be exhibited explicitly.

In section 8, we reinforce some of our conclusions by studying a string-theory realisation of the β-deformed theory. As mentioned above, this involves N D3 branes in the presence of background RR fields. The Higgs branch $\mathcal{H}_m$ corresponds to a configuration in which the D3-branes are polarized into m D5 branes wrapped on a two-dimensional torus of radii $r \sim 2\pi\alpha'\mathrm{v}$. The presence of N units of D3 brane charge leads to non-commutativity on the D5 brane world-volume. We check that the low-energy theory on the D5 world-volume precisely matches the six-dimensional effective action obtained in section 4 by field theory methods.

The string theory construction also provides a nice picture of the electric and magnetic confinement mechanisms discussed above. The magnetic flux tube is realised as a bound-state of a D-string with the D5 branes. This picture confirms the identification between six-dimensional instanton strings and four-dimensional magnetic flux tubes described above. The $\mathrm{SL}(2,\mathbb{Z})$ duality of the β-deformed theory corresponds to the S-duality of IIB string theory. Performing an S-duality transformation on the toroidally wrapped D5 branes leads to a configuration of m NS5-branes wrapped on the same torus. This is the string theory realisation of the theory on the magnetic Higgs branch denoted $\tilde{\mathcal{H}}_m$ above. The electric flux tubes of the confining phase are realised as fundamental strings bound to the NS5-branes. This is very similar to the discussion of confinement in the $\mathcal{N} = 1^*$ theory given by Polchinski and Strassler [2].

So far we have mainly discussed the semiclassical regime where the length-scale set by the six-dimensional gauge coupling is much smaller than the lattice spacing. As for any lattice model, an obvious question is whether the theory has an interacting continuum limit. It turns out that this is closely related to the issue raised above of the possible existence of regimes with light strings. In section 9, we present a preliminary discussion of this question. We propose a one-parameter family of continuum limits. Via S-duality these limits can be interpreted as large-N limits of the confining phase theory. We identify the resulting continuum theory as a decoupled theory on the NS5 branes appearing in the brane construction of section 8. In one special case we obtain commutative Little String Theory in six non-compact dimensions. In addition, we identify an alternative limit analogous to the one considered in [17], which yields commutative Little String Theory compactified to four dimensions on a torus of fixed size. Strikingly, the limit in question is simply a 't Hooft limit of the β-deformed theory in a fixed vacuum state on its confining branch $\tilde{\mathcal{H}}_m$. Finally, we also identify field theory limits which are dual to double-scaled Little String

JHEP08(2004)043

Theory [55]. This case is particularly interesting as the full spectrum of the theory can be calculated explicitly in a particular regime of parameters. The results discussed in section 9 are presented in more detail in [1].

The rest of the paper is organised as follows. In section 2, we introduce the β-deformed theory and discuss its conformal invariance and S-duality. In section 3, we analyse the classical vacuum structure of the theory. Section 4 is devoted to the classical equivalence between the β-deformed theory and the six-dimensional lattice theory discussed above. In section 5, we derive the classical low energy effective action and comment on quantum corrections. Section 6 discusses the classical physics of BPS strings on the Higgs branch. Section 7 presents evidence for the existence of the S-dual confining branches. In section 8 we present a string theory construction of the β-deformed theory. The continuum limit and its relation to Little String Theory is considered in section 9.

2. The $\mathcal{N}=4$ theory and its deformations

We start by considering $\mathcal{N}=4$ SUSY Yang-Mills theory with gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ and complexified coupling constant $\tau = 4\pi i/g^2 + \theta/2\pi$. The theory contains a single vector multiplet of $\mathcal{N}=4$ supersymmetry which includes the gauge field together with four species of adjoint Weyl fermions and six real adjoint scalars, which transform in the **4** and **6** of the SU(4) R-symmetry group respectively. This theory has three remarkable but, by now, well-established properties:

1. The β-function of the gauge coupling, g^2, is exactly zero. The theory is finite and, in the absence of scalar vacuum expectation values (VEVs), it is conformally invariant [21].

2. The theory has an exact S-duality [22] which acts on the complexified coupling as $\tau \to (a\tau + b)/(c\tau + d)$ where the integers a, b, c and d, with $ad - bc = 1$ define an element of the modular group $\mathrm{SL}(2,\mathbb{Z})$.

3. In the 't Hooft large-N limit, $N \to \infty$ with $\lambda = g^2 N$ held fixed, the $\mathrm{SU}(N)$ theory is dual to type-IIB superstring theory on $AdS_5 \times S^5$ of radius $\mathcal{R}$, with $\mathcal{R}^4/\alpha'^2 = \lambda$ [23]. The dual theory can be reliably approximated by IIB supergravity for $\lambda \gg 1$.

The basic objects in the conformally invariant $\mathcal{N}=4$ theory are operators, $\hat{O}$ of definite scaling dimension Δ. As in any conformal theory, a natural operation is to deform the theory by adding one of these operators to the lagrangian;

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \mu \hat{O}\,. \tag{2.1}$$

In four-dimensions, the operator is irrelevant if $\Delta > 4$. In this case the perturbation grows in the UV and extra information is needed to define the theory at short-distances. On the other hand if $\Delta < 4$, the operator is relevant and the perturbation grows in the IR. In this case the theory typically flows to another CFT with lower central charge or a massive theory at low energies. When $\Delta = 4$, then $\hat{O}$ is marginal, and the deformation leads

JHEP08(2004)043

to a new conformal theory with the same central charge. In this case the corresponding coupling, μ, is dimensionless with vanishing β-function and it can be chosen to be small at all energy scales. Only in this case, does it make sense to treat the deformation as a small perturbation around the $\mathcal{N}=4$ theory.

In the following we will focus on a deformation of the $\mathcal{N}=4$ theory with gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ which is known to be exactly marginal and also preserves $\mathcal{N}=1$ supersymmetry. In the language of $\mathcal{N}=1$ supersymmetry, the $\mathcal{N}=4$ theory contains a single vector multiplet V and three chiral multiplets Φ_i $i=1,2,3$ in the adjoint representation of the gauge group. The classical superpotential for the chiral multiplets is,

$$\mathcal{W}_{\mathcal{N}=4} = i\,\mathrm{Tr}_N\left(\Phi_1[\Phi_2,\Phi_3]\right). \tag{2.2}$$

The theory we will study in this paper is obtained by deforming the commutator which appears in the classical superpotential. Specifically we have,

$$\mathcal{W} = i\kappa\,\mathrm{Tr}_N\left(\Phi_1[\Phi_2,\Phi_3]_\beta\right) \tag{2.3}$$

where,

$$[\Phi_i,\Phi_j]_\beta = \exp\left(i\frac{\beta}{2}\right)\Phi_i\Phi_j - \exp\left(-i\frac{\beta}{2}\right)\Phi_j\Phi_i\,. \tag{2.4}$$

We have also introduced a parameter κ which controls the normalization of the scalar potential relative to the kinetic terms in the lagrangian. The original $\mathcal{N}=4$ theory is recovered for $\kappa=1$ and $\beta=0$.

For non-zero values of the deformation parameter β, the SU(4) R-symmetry of the $\mathcal{N}=4$ theory is broken down to $\mathrm{U}(1)^3$. Each U(1) rotates the phase of one of the three complex chiral superfields. The deformation also leads to important differences between the $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ theories. In the undeformed $\mathrm{U}(N)$ theory the photon of the central U(1) subgroup is part of an $\mathcal{N}=4$ multiplet which also includes degrees of freedom corresponding to the traces of the adjoint scalar and fermion fields. The whole multiplet is completely decoupled from the traceless $\mathrm{SU}(N)$ fields. For $\beta\neq 0$, the central photon and its $\mathcal{N}=1$ superpartner still decouple, but the chiral multiplets $\mathrm{Tr}_N\,\Phi_i$ for $i=1,2,3$ do not. The $\mathrm{U}(N)$ theory is equivalent to the $\mathrm{SU}(N)$ theory with couplings of order $|\beta|$ to these extra chiral multiplets together with a decoupled U(1) vector multiplet.

The deformation of the superpotential involves two linearly independent operators,

$$\hat{O}_1 = \mathrm{Tr}_N\left(i\Phi_1[\Phi_2,\Phi_3]\right) \qquad \hat{O}_2 = \mathrm{Tr}_N\left(i\Phi_1\{\Phi_2,\Phi_3\}\right) \tag{2.5}$$

At linear order in the deformation parameters β and $\kappa-1$, the resulting modification of the $\mathcal{N}=4$ lagrangian involves the the supersymmetric descendents of these operators obtained by acting with two supercharges. Schematically we have,

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \frac{1}{g^2}(\kappa-1)\,Q^2\hat{O}_1 + \frac{1}{g^2}\beta\,Q^2\hat{O}_2 + \text{h.c.} \tag{2.6}$$

Note that we have chosen to normalize the fields so that every term in the lagrangian has an overall prefactor of $1/g^2$.

Both operators appearing in (2.6) have classical mass dimension four. However, the corresponding couplings are only exactly marginal if the operators do not get anomalous dimensions. At linear order in the deformation parameters, the dimensions of the operators appearing in (2.6) can be replaced by their $\mathcal{N}=4$ values. In the $\mathcal{N}=4$ theory, the operator $\hat{O}_2$ is part of a chiral primary multiplet. The full multiplet consists of symmetric, traceless, third-rank tensors of SO(6) of the form $Tr_N(\phi^{\{I}\phi^J\phi^{K\}})$, where ϕ^I with $I=1,2,\dots 6$ are the six real adjoint fields of the $\mathcal{N}=4$ theory[6] [24]. This means that neither $\hat{O}_2$ nor its descendent $Q^2\hat{O}_2$ acquire an anomalous dimension for any value of the $\mathcal{N}=4$ coupling τ. On the other hand, $\hat{O}_1$, which involves an anti-symmetrized trace, is part of a non-chiral multiplet and its scaling dimension is not protected. In fact, the AdS/CFT correspondence relates these operators to stringy excitations which decouple in the SUGRA limit $\lambda\to\infty$. We therefore expect $\hat{O}_1$ and its descendent $Q^2\hat{O}_1$ to acquire large positive anomalous dimensions of order $\lambda^{\frac{1}{4}}$ at strong coupling.

The argument given above suffices to show that the coupling β is exactly marginal at linear order in the deformation parameters while $\kappa-1$ is not.[7] To show that a marginal coupling survives beyond linear order is much harder because we can no longer rely on $\mathcal{N}=4$ supersymmetry. Remarkably, it is still possible to make progress using the much weaker constraints of $\mathcal{N}=1$ supersymmetry. Leigh and Strassler [3] studied the exact NSVZ β-functions for the three couplings τ, β and κ. The existence of a conformal fixed-point requires that all three β-functions vanish. This leads to three complex equations for three unknowns which typically has at most isolated solutions. However, in the present case, the authors of [3] found that two of the three equations are redundant allowing for a two-complex parameter families of solutions. Specifically they found that the the theory has a critical surface in coupling constant space defined by $\kappa=\kappa_{cr}[\tau,\beta]$ which includes the $\mathcal{N}=4$ point $\beta=0$, $\kappa=1$. The explicit form of the function $\kappa_{cr}[\tau,\beta]$ is unknown beyond one-loop. However, for $\beta=0$ and $\kappa=1$ we must recover the $\mathcal{N}=4$ theory. The linearized analysis described above indicates that near this point the two marginal couplings are simply τ and β. Thus we have $\kappa_{cr}[\tau,\beta]=1+O(|\beta|^2)$.

The resulting surface of complex dimension two is parametrized by complex couplings which are exactly marginal to all orders in perturbation theory and also non-perturbatively. It therefore corresponds to a two-parameter family of $\mathcal{N}=1$ superconformal theories. As above, we will refer to these collectively as the β-deformed theory. In the following, we will rely heavily on the fact that the the theory has a smooth $\beta\to 0$ limit in which $\mathcal{N}=4$ supersymmetry is recovered. Note that one would not expect the corresponding limit to exist for a relevant deformation of the $\mathcal{N}=4$ theory. The β-deformation is not quite the most general possible marginal deformation of the $\mathcal{N}=4$ theory. It is also possible to add a term of the form $\rho Tr_N(\Phi_1^3+\Phi_2^3+\Phi_3^3)$ to the superpotential leading to a three complex dimensional critical surface $\kappa=\kappa_{cr}[\tau,\beta,\rho]$. We will set $\rho=0$ from now on.

[6] The complex adjoint scalars Φ_i, $i=1,2,3$, appearing in the $\mathcal{N}=1$ formulation of the theory can be written as $\Phi_i=\phi_i+i\phi_{i+3}$.

[7] The corresponding anomalous dimension is positive definite because of the Bogomol'nyi bound, and hence the coupling $\kappa-1$ is marginally irrelevant.

When the 't Hooft coupling is small, which corresponds to $g^2 \ll 1/N$, we can study the β deformation of the $\mathcal{N} = 4$ theory using semiclassical methods. In the following (and in [1]), we will also be interested in the behaviour of the deformed theory at strong coupling. When $1/N \ll g^2 \ll 1$, the undeformed $\mathcal{N} = 4$ theory with gauge group SU(N) is well approximated by IIB supergravity on $AdS_5 \times S^5$. An obvious question is how the β-deformation is realised on the IIB side of the correspondence. As above, it is easiest to work to linear order in the deformation parameters where the exactly marginal coupling is β itself. The corresponding operator Q^2O_2 is a supersymmetric decendent of a chiral primary and is therefore mapped to a supergravity field in AdS_5.

The field in question is a particular S^5 spherical harmonic of the the IIB complexified three-form field strength,

$$G_{(3)} = F_{(3)} - \tau H_{(3)} \tag{2.7}$$

which transforms in the **45** of SU(4) $\simeq$ SO(6). Turning on the β-deformation corresponds to introducing a source for this mode on the boundary of AdS_5. The source results in a small deformation of the $AdS_5 \times S^5$ background which can be constructed by solving the supergravity field equations order by order in $|\beta|$. This program was carried out to second order in the deformation in [13]. This is in marked contrast to the case of a relevant deformation, where the corresponding boundary source leads to a deformation of the geometry which grows in the interior of AdS_5 and cannot be treated perturbatively. As usual, the dual supergravity description becomes exact in the large-N limit with g^2 fixed. The existence of a smooth supergravity description of the β-deformation will be important for what follows because it indicates that the $\beta \to 0$ limit commutes with this large-N limit.

Another regime in which we can hope to make progress is that of ultra-strong coupling $g^2 \gg N$. In the undeformed case, the ultra strongly-coupled $\mathcal{N} = 4$ theory is related to the weakly-coupled theory by an electric-magnetic duality transformation. We can also use S-duality to map an $\mathcal{N} = 4$ theory with $1 \ll g^2 \ll N$, to one with $1/N \ll g^2 \ll 1$ which is well described by IIB supergravity. Thus we need to understand how the SL$(2,\mathbb{Z})$ duality properties of the $\mathcal{N} = 4$ theory extend to the β-deformed case. Once again it is easiest to make progress at linear order in the deformation, where we are perturbing the $\mathcal{N} = 4$ theory by adding the term $(\beta/g^2)Q^2\hat{O}_2$ (and its hermitian conjugate) to the lagrangian. In general we do not expect the new term in the lagrangian to be invariant under the S-duality of the $\mathcal{N} = 4$ theory. However, we will see that S-duality can be restored by assigning an appropriate transformation property to the complex coupling, β.

A key property of the chiral primary operators of the $\mathcal{N} = 4$ theory (and their supersymmetric descendents) is that they have well-defined transformation properties under SL$(2,\mathbb{Z})$ [25] (see also [26]). In particular they transform as non-holomorphic modular forms. A modular form F of weight $(w, \bar{w})$ transforms as,

$$F \to (c\tau + d)^w (c\bar{\tau} + d)^{\bar{w}}\, F \tag{2.8}$$

under the SL$(2,\mathbb{Z})$ transformation defined by integers a, b, c, d with $ad - bc = 1$. The basic rule is that a chiral primary of mass dimension d obtained as a symmetrized trace of a product of d scalars has weight $(\frac{d}{2},\frac{d}{2})$. Thus $\hat{O}_2$ has weight $(\frac{3}{2},\frac{3}{2})$. Following the conventions of [26], the left-handed supercharges Q_α have weights $(\frac{1}{4},-\frac{1}{4})$ and the gauge coupling g^2 has weight $(1,1)$. Thus we see that the S-duality of the $\mathcal{N}=4$ theory can be preserved in the presence of the deformation provided we assign β holomorphic weight $(-1,0)$. In other words, at linear order in the deformation, we have the $\mathrm{SL}(2,\mathbb{Z})$ transformation property,

$$\beta \to \frac{\beta}{(c\tau+d)} \tag{2.9}$$

So far our considerations are only valid for an infinitessimal deformation of the $\mathcal{N}=4$ theory. Beyond linear order, the β-deformation involves other operators beside $Q^2\hat{O}_2$ and we cannot even assume that this operator retains the same transformation properties it has in the $\mathcal{N}=4$ theory. As in the discussion of the existence of exactly marginal operators given above, we must now rely only on the much weaker constraints of $\mathcal{N}=1$ supersymmetry. Despite this, we now have strong evidence that S-duality does extend to all values of the marginal couplings. As we now review, one can even specify the exact transformation properties of these couplings.

As we have seen, the β-deformation of the $\mathcal{N}=4$ theory leads to a two-parameter family of superconformal field theories. We can also further deform the theory by adding relevant operators. The resulting model then flows to the β-deformed conformal theory in the UV, but its behaviour will be different in the IR. In [15], a particular relevant deformation was introduced by adding an additional term,

$$\Delta\mathcal{W} = \mathrm{Tr}_N\left[M\Phi_1\Phi_2 + \mu\Phi_3^2\right] \tag{2.10}$$

to the superpotential (2.3) of the β-deformed theory. For $\beta=0$, the resulting theory is equivalent (by a global SO(3) rotation) to the standard $\mathcal{N}=1^*$ deformation of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. The vacuum structure of the theory is well understood in this case [27, 28, 29] and exhibits an exact $\mathrm{SL}(2,\mathbb{Z})$ duality, acting on the complex coupling τ, which is inherited from the $\mathcal{N}=4$ theory.

The $\mathcal{N}=1^*$ theory has a variety of vacuum states in different massless and massive phases. The massive vacua are realised in various Higgs and confining phases associated with the condensation of different electric and magnetically charged states. As the states which condense transform non-trivially under electric-magnetic duality, $\mathrm{SL}(2,\mathbb{Z})$ acts by permuting different vacuum states [28]. For example the 'S' generator of $\mathrm{SL}(2,\mathbb{Z})$ interchanges vacua in the Higgs and confining phases. The full action of $\mathrm{SL}(2,\mathbb{Z})$ on the $\mathcal{N}=1^*$ vacua is determined by the effective superpotential of the theory derived in [29]. The superpotential also determines the condensates of $\mathcal{N}=1$ chiral operators in each vacuum. For example, in any supersymmetric vacuum state we have,

$$\langle \mathrm{Tr}_N\,\Phi_3^2\rangle = \frac{\partial}{\partial\mu}\langle\mathcal{W}_{\mathrm{eff}}\rangle\,. \tag{2.11}$$

Similar results for other condensates can also be obtained [30]. The resulting formulae are consistent with the known modular weights of the same operators in the $\mathcal{N}=4$ theory. For example, the vacuum expectation value $\langle\mathrm{Tr}_N\Phi_3^2\rangle$ transforms like a modular form of weight $(1,1)$ modulo the vacuum permutations described above [26].

Our ability to calculate exactly in the $\mathcal{N}=1^*$ theory is limited to holomorphic quantities like the chiral condensates described above. Thus we are only able to check the proposed $\mathrm{SL}(2,\mathbb{Z})$ duality for these special quantities. However, the $\mathcal{N}=1^*$ theory flows in the UV to $\mathcal{N}=4$ theory where the corresponding duality is believed to be exact. As both the UV behaviour of the $\mathcal{N}=1^*$ theory and its IR vacuum structure are $\mathrm{SL}(2,\mathbb{Z})$ invariant, it is natural to conjecture that the duality is an exact property of the theory valid at all length-scales. Futher support for this view-point comes from the AdS/CFT correspondence where the $\mathrm{SL}(2,\mathbb{Z})$ duality of the $\mathcal{N}=4$ theory is mapped to the S-duality of IIB string theory which is also believed to be exact. AdS duals of the massive $\mathcal{N}=1^*$ vacua were constructed in [2]. In particular, different vacua of the $\mathcal{N}=1^*$ theory were realised as different string backgrounds permuted by the exact S-duality of the IIB theory.

In [15], we presented evidence that a similar story is realised in the β-deformed case. Specifically, the exact effective superpotential of both the $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ theories was determined via the Dijkgraaf-Vafa proceedure [31] by solving a related matrix model. In both case the results revealed an $\mathrm{SL}(2,\mathbb{Z})$ duality acting on a renormalized gauge coupling,

$$\tau_R=\frac{4\pi i}{g_R^2}+\frac{\theta_R}{2\pi}=\tau+\frac{iN}{\pi}\log\kappa \tag{2.12}$$

and also on the deformation parameters as

$$\tau_R\to\frac{a\tau_R+b}{c\tau_R+d}\qquad \beta\to\frac{\beta}{c\tau_R+d}\qquad \kappa^2\sin\beta\to\frac{\kappa^2\sin\beta}{c\tau_R+d} \tag{2.13}$$

although there are some complications for the $\mathrm{U}(N)$ theory which we discuss in appendix A. It is easy to check this reduces to standard S-duality at the $\mathcal{N}=4$ point $\beta=0$, $\kappa=1$ and also reproduces the result (2.9) of the linearized analysis, namely that β has modular weight $(-1,0)$. The results of [15] show that chiral operators transform with the same modular weights as they do in the $\mathcal{N}=4$ theory. Like the S-duality of the $\mathcal{N}=4$ theory, we will assume that (2.13) is an exact duality of the β-deformed theory for all values of the parameters.

3. Classical vacuum structure

In this section we will study the classical vacuum structure of the $\mathrm{U}(N)$ theory and also make some comments about quantum corrections. The F- and D-flatness conditions read,

$$[\Phi_1,\Phi_2]_\beta=[\Phi_2,\Phi_3]_\beta=[\Phi_3,\Phi_1]_\beta=0 \tag{3.1}$$

and

$$\sum_{i=1}^{3}[\Phi_i,\Phi_i^\dagger]=0 \tag{3.2}$$

respectively. For the $\mathcal{N}=4$ case, $\beta=0$, the deformed commutators appearing in the F-term constraint revert to ordinary ones. In this case the vacuum equations are solved by diagonalizing each of the three complex scalars,

$$\langle\Phi_i\rangle=\mathrm{Diag}\left[\lambda_1^{(i)},\lambda_2^{(i)},\ldots,\lambda_N^{(i)}\right]. \tag{3.3}$$

The $3N^2$ complex eigenvalues $\lambda_a^{(i)}$, for $i = 1, 2, 3$ and $a = 1, 2, \ldots, N$, are unconstrained. After taking into account the Weyl group, we recover the familiar Coulomb branch of the $\mathcal{N} = 4$ theory. On this branch the $\mathrm{U}(N)$ gauge symmetry is spontaneously broken down to its Cartan subalgebra $\mathrm{U}(1)^N$ and he vacuum manifold is the symmetric product $\mathrm{Sym}_N\, \mathbb{C}^3$.

Introducing a generic, non-zero value of β changes things considerably. The F-flatness conditions are no longer solved by arbitrary diagonal matrices (3.3). For each value of the Cartan index $a \in \{1, 2, \ldots, N\}$, at most one of the three eigenvalues, $\lambda_a^{(1)}$, $\lambda_a^{(2)}$ and $\lambda_a^{(3)}$ can be non-zero. In the simplest case of gauge group $\mathrm{U}(1)$, the Coulomb branch of the $\mathcal{N} = 4$ theory, which has three complex dimensions, is partially lifted leaving three complex lines which intersect at the origin. For $G = \mathrm{U}(N)$ with $N > 1$, the Coulomb branch is formed by taking an N-fold symmetric product in the usual way.

If $\beta/2\pi$ is not a rational real number, the Coulomb branch vacua described above are the only classical supersymmetric groundstates for any value of N. To illustrate the new possibilities which occur at rational values of $\beta/2\pi$ we will focus on a particular one-parameter family of vacua on the Coulomb branch. We choose $\langle\Phi_1\rangle = \alpha_1 U_{(N)}$ and $\langle\Phi_2\rangle = \langle\Phi_3\rangle = 0$, where α_1 is a complex number and we define the $N \times N$ 'clock' matrix,

$$U_N = \mathrm{Diag}\left[\omega_N, \omega_N^2, \ldots, \omega_N^{N-1}, \omega_N^N\right] \tag{3.4}$$

where $\omega_N = \exp(2\pi i/N)$ is an N'th root of unity. We denote this one-dimensional complex submanifold of the Coulomb branch $\mathcal{C}_1$. Starting at the $\mathcal{N} = 4$ point, $\beta = 0$, the adjoint Higgs mechanism yields the familiar spectrum of $W^{\pm}$ gauge bosons and their superpartners. These states lie in N^2 BPS-saturated vector multiplets of $\mathcal{N} = 4$ supersymmetry with masses,

$$M^{(ab)} = 2|\alpha_1| \sin\left(\frac{\pi}{N}|a-b|\right) \tag{3.5}$$

for $a, b = 1, 2, \ldots, N$. As these states are BPS saturated, this mass formula is exact. Turning on a non-zero value of β breaks $\mathcal{N} = 4$ supersymmetry and each $\mathcal{N} = 4$ multiplet splits into $\mathcal{N} = 1$ multiplets. In particular each massive BPS-saturated vector multiplet of $\mathcal{N} = 4$ SUSY splits into a massive vector multiplet of $\mathcal{N} = 1$ SUSY and two additional massive chiral multiplets. While the vector multiplet masses are unaffected, the chiral multiplet masses depend explicitly on β as,

$$M_{\pm}^{(ab)} = 2|\alpha_1| \sin\left(\left|\frac{\pi}{N}(a-b) \pm \frac{\beta}{2}\right|\right) \tag{3.6}$$

The unbroken $\mathcal{N} = 1$ supersymmetry algebra does not have a central charge and so none of these states are BPS. This means that, for $\beta \neq 0$, the classical mass formulae (3.5), (3.6) can recieve quantum corrections. As the masses are protected in the $\mathcal{N} = 4$ limit, we expect that these corrections are small for $|\beta| \ll 1$.

The classical mass formula (3.6) indicates that new massless states occur when $\beta = 2\pi l/N$ where l is an integer. The occurence of these states reflects the presence of a new branch of vacua. We begin by focussing on the case $l = 1$. When $\beta = 2\pi/N$, the F-term equations have non-trivial solutions involving the 'clock' matrix $U_{(N)}$ introduced above and the $N \times N$ 'shift' matrix $V_{(N)}$ defined by,

$$\begin{aligned} \left(V_{(N)}\right)_{ab} &= 1 \qquad \text{if } b = a+1 \bmod N \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{3.7}$$

These $N \times N$ matrices are both unitary and satisfy,

$$\left[U_{(N)}, V_{(N)}\right]_{\frac{2\pi}{N}} = \sqrt{\omega_N} U_{(N)} V_{(N)} - \sqrt{\bar{\omega}_N} V_{(N)} U_{(N)} = 0\,. \tag{3.8}$$

Using this relation we find a three complex parameter branch of vacua. Up to gauge transformations the scalar VEVs can be chosen as,

$$\langle \Phi_1 \rangle = \alpha_1 U_{(N)} \qquad \langle \Phi_2 \rangle = \alpha_2 V_{(N)} \qquad \langle \Phi_3 \rangle = \alpha_3 W_{(N)} \tag{3.9}$$

where $W_N = V_{(N)}^\dagger U_{(N)}^\dagger$. Here α_1, α_2 and α_3 are complex numbers. At a generic point on this branch of vacua, the U(N) gauge symmetry is broken to its central U(1) subgroup and the U$(1)^3$ R-symmetry is broken to $\mathbb{Z}_N \times \mathbb{Z}_N$. As the expectation values of the adjoint scalars are traceless, this branch is also present in the SU(N) theory. In the SU(N) case the gauge group is completely broken. For future convenience, we will refer to this branch as the Higgs branch $\mathcal{H}_1$.

There is still some redundancy our parametrization of the Higgs branch. Specifically, the U(N) gauge transformations;

$$\Phi_i \to \Gamma_j \Phi_i \Gamma_j^\dagger \tag{3.10}$$

for $i = 1, 2, 3$ and $j = 1, 2$ with $\Gamma_1 = U_{(N)}$ and $\Gamma_2 = V_{(N)}$ lead to discrete phase rotations,

$$\begin{aligned} \Gamma_1: &\quad \alpha_1 \to \alpha_1 \qquad && \alpha_2 \to \bar{\omega}_N \alpha_2 \qquad && \alpha_3 \to \omega_N \alpha_3 \\ \Gamma_2: &\quad \alpha_1 \to \omega_N \alpha_1 \qquad && \alpha_2 \to \alpha_2 \qquad && \alpha_3 \to \bar{\omega}_N \alpha_3\,. \end{aligned} \tag{3.11}$$

Taking into account these identifications the Higgs branch $\mathcal{H}_1$ is the complex orbifold $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$. This Higgs branch intersects the Coulomb branch on the submanifold $\mathcal{C}_1$ defined above when $\alpha_2 = \alpha_3 = 0$. The intersection is a fixed-line of the orbifold on which Γ_1 acts trivially. In addition, $\mathcal{H}_1$ intersects other components of the Coulomb branch when $\alpha_1 = \alpha_3 = 0$, $\alpha_2 \neq 0$ and when $\alpha_1 = \alpha_2 = 0$, $\alpha_3 \neq 0$.

It will be useful to have a description of the vacuum structure in terms of gauge invariant variables. Hence we define the gauge invariant chiral operators,

$$u_{(k_1,k_2,k_3)} = \frac{1}{N} \operatorname{Tr}_N \left[\Phi_1^{k_1} \Phi_2^{k_2} \Phi_3^{k_3}\right]. \tag{3.12}$$

On the Higgs branch $\mathcal{H}_1$ discussed above, these VEVs of these operators vanish unless *either*

$$(k_1, k_2, k_3) = (0, 0, 0) \bmod N \qquad \text{or} \qquad k_1 = k_2 = k_3 \bmod N \tag{3.13}$$

in which case,

$$\langle u_{(k_1,k_2,k_3)} \rangle = \exp(i\nu_{(k_1,k_2,k_3)}) \alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \tag{3.14}$$

where $\exp(i\nu_{(k_1,k_2,k_3)})$ is an unimportant phase.

JHEP08(2004)043

On this branch, the chiral ring is generated by four non-zero elements, $x = u_{(N,0,0)}$, $y = u_{(0,N,0)}$ $z = u_{(0,0,N)}$ and $w = u_{(1,1,1)}$ subject to the relation

$$w^N = xyz\,. \tag{3.15}$$

Equivalently we can think of x, y, z and w as gauge-invariant complex coordinates on the Higgs branch $\mathcal{H}_1$. The equation (3.15) defines the complex orbifold $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$.

The massless modes on the Higgs branch $\mathcal{H}_1$ are the scalar moduli α_i, $i = 1, 2, 3$ and the photon of the central U(1). Each of these bosonic fields is paired with a massless Weyl fermion by the unbroken $\mathcal{N} = 1$ supersymmetry. These fields are free at low energies and the effective action is precisely that of an $\mathcal{N} = 4$ supersymmetric gauge theory with gauge group $H = \mathrm{U}(1)$. Note that this enhancement of supersymmetry in the IR theory is special to the U(N) theory and does not occur for gauge group SU(N). The complex coupling of the low-energy theory is related to that of the original theory as $\tau_H = N\tau$. Equivalently the low-energy U(1) gauge coupling is $g_H^2 = g^2/N$. As τ_H is an F-term coupling any corrections to the classical relation $\tau_H = N\tau$ must be holomorphic in τ. The results of [15], show that the exact low energy coupling is $N\tau_R$, where the renormalised coupling τ_R is given in Eq (2.12) above.

The β-deformed lagrangian has $\mathcal{N} = 1$ superconformal invariance and $\mathrm{U}(1)^3$ R-symmetry. These symmetries are spontaneously broken on the Higgs branch and some of the massless scalars discussed above correspond to massless Goldstone bosons. The remaining massless modes lie in $\mathcal{N} = 1$ multiplets with the Goldstone modes. As the symmetries in question are non-anomalous, all these fields remain massless in the full quantum theory. Thus we deduce that the Higgs branch $\mathcal{H}_1$ cannot be lifted by quantum effects. The equation (3.15), which defines the Higgs branch as a complex manifold is also protected from quantum correstions. For example, additional terms in (3.15) which resolve the orbifold singularity are ruled out by the non-anomalous R-symmetries of the theory.

In the following, we will also be interested in more general Higgs branches of the theory with a larger unbroken gauge group. These occur when the rank N has a non-trivial divisor. Thus we have $N = mn$ for some integers m and n. If the deformation parameter takes the value $\beta = 2\pi/n$ we find a three complex parameter branch on which the scalar VEVs can be chosen as,

$$\langle\Phi_1\rangle = \alpha_1 I_{(m)} \otimes U_{(n)} \qquad \langle\Phi_2\rangle = \alpha_2 I_{(m)} \otimes V_{(n)} \qquad \langle\Phi_3\rangle = \alpha_3 I_{(m)} \otimes W_{(n)} \tag{3.16}$$

where $I_{(m)}$ is the $m \times m$ unit matrix. Identifications generalising (3.10), lead to a Higgs branch $\mathcal{H}_m$ which is the complex orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. On this branch, the VEVs of the operators $u_{(k_1,k_2,k_3)}$ introduced above are given by replacing N by n in (3.13) and (3.14). As for $\mathcal{H}_1$, the flat directions are protected by Goldstone's theorem and persist for all values of the gauge coupling. The low-energy theory on this branch is $\mathcal{N} = 4$ SUSY Yang-Mills theory with gauge group $H = \mathrm{U}(m)$ with complexified coupling $\tau_H = \tau/n$. Equivalently, the low-energy gauge coupling is $g_H^2 = g^2/n$.

The low-energy theory on $\mathcal{H}_m$ is an $\mathcal{N}=4$ theory at its conformal point. This theory also has a Coulomb branch and it is quite straightforward to find the corresponding vacua of the original $\mathrm{U}(N)$ theory. They are obtained by choosing the scalar VEVs as,

$$\langle\Phi_1\rangle = \alpha_1\Lambda^{(1)}\otimes U_{(n)} \qquad \langle\Phi_2\rangle = \alpha_2\Lambda^{(2)}\otimes V_{(n)} \qquad \langle\Phi_3\rangle = \alpha_3\Lambda^{(3)}\otimes W_{(n)} \tag{3.17}$$

where $\Lambda^{(i)}$ $i=1,2,3$ are three diagonal $m\times m$ matrices,

$$\Lambda^{(i)} = \mathrm{Diag}\left[\lambda_1^{(i)},\lambda_2^{(i)},\ldots,\lambda_m^{(i)}\right]. \tag{3.18}$$

When these eigenvalues are distinct the unbroken gauge group is $\mathrm{U}(1)^m$. This corresponds to the Coulomb phase of the low energy $\mathrm{U}(m)$ theory described above. Apart from the Weyl group of $\mathrm{U}(m)$ which permutes the eigenvalues we must also identify vacua which are related by discrete gauge transformations of the form (3.10). In the present case there are $2m$ such transformations generated by matrices $\Gamma_{1r}=P_{(r)}\otimes U_{(n)}$ and $\Gamma_{2r}=P_{(r)}\otimes V_{(n)}$ for $r=1,2,\ldots,m$ where $P^{(r)}_{st}=\delta_{sr}\delta_{tr}$. After dividing out by all of these transformations we find a vacuum manifold which is the symmetric product of m copies of the orbifold $\mathbb{C}^3/\mathbb{Z}_n\times\mathbb{Z}_n$.

The appearance of the complex orbifold $\mathbb{C}^3/\mathbb{Z}_n\times\mathbb{Z}_n$ naturally leads us to mention the standard string theory realisation of the β-deformed theory. As demonstrated in [5, 6], the $\mathrm{U}(N)$ theory with $N=mn$ and $\beta=2\pi/n$ arises on the world volume of m D3 branes placed at a $\mathbb{C}^3/\mathbb{Z}_n\times\mathbb{Z}_n$ singularity with a single unit of *discrete torsion*. The discrete torsion corresponds to the weighting of the twisted sectors of the orbifold theory with the phase $\exp(i\beta)=\exp(2\pi i/n)$. The Higgs branches $\mathcal{H}_m$ considered above corresponds to the position of m coincident D3-branes in the orbifold space. As usual, moving onto the Coulomb branch of the low energy theory corresponds to separating the branes. The resulting moduli space of vacua described in the previous paragraph is the symmetric product of m copies of $\mathbb{C}^3/\mathbb{Z}_n\times\mathbb{Z}_n$. This corresponds to the full configuration space of m D3 branes in the orbifold. This orbifold construction of the β-deformed theory is important for the discussion of deconstruction given in [10]. We will also discuss a quite different stringy realisation of the Higgs branch in section 8.

4. Classical deconstruction

In this section we will discuss the β-deformed theory on the Higgs branch $\mathcal{H}_m$ and its Kaluza-Klein interpretation in terms of a six-dimensional lattice gauge theory. We will consider in turn the spectrum and the interactions. For simplicity we will restrict our attention to the submanifold $\mathcal{H}_m$ where only two of the three complex scalar fields acquire VEVs. Thus we set $\alpha_3=0$ in (3.16). Correspondingly, we will adopt a convention where indices i, j and k take values 1 or 2 only. Throughout this section we will restrict our attention to the case of gauge group $\mathrm{U}(N)$.

4.1 The spectrum

To find the full classical spectrum in these vacua, we must expand the fields in fluctuations around their vacuum values and diagonalize the resulting quadratic terms in the action. To accomplish this we need to define a convenient basis which we now describe.

All the fields of the theory are $N \times N$ matrices and we will think of them as elements of the algebra $gl(N,\mathbb{C})$. It will also be useful to use the decompose $gl(N,\mathbb{C})$ as a tensor product,

$$gl(N,\mathbb{C}) \equiv gl(m,\mathbb{C}) \otimes gl(n,\mathbb{C})\,. \tag{4.1}$$

We now choose the following set of n^2 basis elements for $gl(n,\mathbb{C})$ labelled by a vector $\vec{l} = (l_1, l_2)$ with integer components l_1 and l_2:

$$J^{(\vec{l})} = V_{(n)}^{l_1} U_{(n)}^{-l_2}\, \omega_n^{\frac{l_1 l_2}{2}}\,. \tag{4.2}$$

Where $U_{(n)}$ and $V_{(n)}$ are $n \times n$ clock and shift matrices defined as in (3.4) and (3.7) above and $\omega_n = \exp(2\pi i/n)$. By virtue of the relations $U_{(n)}^n = V_{(n)}^n = I_{(n)}$, the integers l_1 and l_2 are only defined modulo n. Equivalently, the vector $\vec{l}$ takes values in $\mathbb{Z}_n^2$. The completeness of this basis means that for any $N \times N$ matrix A we define the expansion,

$$A = \frac{1}{n^2} \sum_{\vec{l} \in \mathbb{Z}_n^2} a^{(\vec{l})} \otimes J^{(\vec{l})} \tag{4.3}$$

where each of the n^2 coefficients $a^{(\vec{l})}$ is itself an $m \times m$ matrix field. For a hermitean matrix field $A(y)$, the additional constraint $a^{(\vec{l})\,\dagger} = a^{(-\vec{l})}$ must be imposed.

We will begin by discussing the spectrum of massive gauge bosons on the Higgs branch $\mathcal{H}_m$. To this end we expand the U(N) gauge field A_μ as in (4.3). It is then easy to check that the corresponding coefficient $a_\mu^{(\vec{l})}$ is a mass eigenstate with eigenvalue,

$$\left(M^{(\vec{l})}\right)^2 = 4|\alpha_1|^2 \sin^2\left(\frac{l_1 \pi}{n}\right) + 4|\alpha_2|^2 \sin^2\left(\frac{l_2 \pi}{n}\right). \tag{4.4}$$

This result is consistent with the Higgs mechanism on $\mathcal{H}_m$ which breaks the gauge group $G = \mathrm{U}(N)$ down to $H = \mathrm{U}(m)$. In particular, the mode $a_\mu^{(\vec{0})}$ corresponds to the m^2 massless gluons of the low energy U(m) gauge theory. The remaining $n^2 - 1$ modes correspond to massive W-bosons and the degeneracy of m^2 states at each mass level reflects the fact that the W-bosons transform in the adjoint of the unbroken U(m).

As the theory has unbroken $\mathcal{N} = 1$ supersymmetry, the full spectrum must consist only of $\mathcal{N} = 1$ multiplets. The on-shell states which appear in each massive multiplet are classified according to their transformation properties under the SU(2) little group. Restricting our attention to on-shell massive states with spin ≤ 1, there are only two types of multiplet which can occur [32]. A massive vector multiplets contains one spin 1 state, two spin 1/2 states and one spin 0 state. A massive chiral multiplet contains one spin 1/2 state and two spin 0 states. The spectrum of massive W-bosons found above must lie in $n^2 - 1$ massive vector multiplets each transforming in the adjoint of U(m). The remaining states in the spectrum, having spins $\leq 1/2$, must be packaged into massive chiral multiplets.

Having identified which $\mathcal{N} = 1$ multiplets occur, the full spectrum can be deduced by diagonalizing the fermion mass terms. As for the gauge fields, this proceeds by expanding each adjoint fermion field as in (4.3). It turns out that the extra chiral multiplets are degenerate with the massive vector multiplets and hence that *all* states in the theory have the masses given by (4.4). For each value of $\vec{l} = (l_1, l_2) \neq \vec{0}$, we have a single massive vector multiplet and two massive chiral multiplets[8] with mass $M^{(\vec{l})}$, each transforming in the adjoint representation of $\mathrm{U}(m)$. Thus the matter content at each mass level is exactly that of a massive BPS-saturated vector multiplet of $\mathcal{N} = 4$ supersymmetry.

The interpretation of the spectrum described above is clearest in the large-n limit. Specifically, as $n \to \infty$, we have an infinite tower of states labelled by two arbitrary integers l_1 and l_2 (which we hold fixed in the limit). In this case we can make the replacement $\sin(l_i\pi/n) \simeq l_i\pi/n$ for $i = 1, 2$ and the mass formula (4.4) becomes,

$$M^{(\vec{l})} = \sqrt{l_1^2\left(\frac{2\pi|\alpha_1|}{n}\right)^2 + l_2^2\left(\frac{2\pi|\alpha_2|}{n}\right)^2} \tag{4.5}$$

with a degeneracy identical to that of a single massive BPS multiplet of $\mathcal{N} = 4$ supersymmetry transforming in the adjoint representation of $\mathrm{U}(m)$ at each non-zero mass level. Remarkably this spectrum is identical to that of a six-dimensional $\mathrm{U}(m)$ gauge theory with non-chiral $\mathcal{N} = (1,1)$ supersymmetry compactified down to four dimensions with SUSY preserving boundary conditions on a rectangular torus of sides,

$$R_1 = \frac{n}{2\pi|\alpha_1|} \qquad R_2 = \frac{n}{2\pi|\alpha_2|} \tag{4.6}$$

The integers l_1 and l_2 appearing in (4.5) correspond to the quantized momenta of each field along the compact dimensions.

At energies far below the compactification scale, $E \ll 1/R_1$, $1/R_2$ the six-dimensional theory reduces to a four-dimensional $\mathcal{N} = 4$ theory with gauge group $\mathrm{U}(m)$ and gauge coupling $G_4^2 = 4\pi^2 R_1 R_2/G_6^2$ where G_6 is the six dimensional gauge coupling. This agrees with the low energy theory on the Higgs branch $\mathcal{H}_m$ provided that we identify,

$$G_6^2 = \frac{g^2 n}{|\alpha_1||\alpha_2|}\,. \tag{4.7}$$

In addition, the conserved momenta of the six-dimensional theory along the compactified directions appear as central charges in the $\mathcal{N} = 4$ supersymmetry algebra of the low-energy theory. Consequently, the massive KK modes form BPS multiplets of the low-energy $\mathcal{N} = 4$ supersymmetry in agreement with the spectrum of the β-deformed theory.

Although we started with a theory with $\mathcal{N} = 1$ supersymmetry in four dimensions, the spectrum we find matches that of a compactified six-dimensional theory with a much larger supersymmetry algebra. In fact, the recovery of sixteen supercharges as $n \to \infty$ is very natural as the deformation parameter $\beta = 2\pi/n$ tends to zero in this limit and we formally recover the $\mathcal{N} = 4$ supersymmetry of the undeformed theory. In this limit, the entire classical spectrum consists of BPS multiplets of $\mathcal{N} = 4$ supersymmetry. As the masses of BPS states are not renormalized in an $\mathcal{N} = 4$ theory, this suggests that the classical mass formula (4.4) becomes exact in the large-n limit. The issue of quantum corrections will be discussed in more detail in section 5.3 below.

[8] To be more precise, the mass formula for the different fermion species differ from (4.4) by replacements of the form $(l_1, l_2) \to (l_1 \pm 1, l_2 \pm 1)$ which leave the complete spectrum invariant.

JHEP08(2004)043

Setting one of the momenta equal to zero (say l_2) the $n \to \infty$ spectrum has the standard form of a Kaluza-Klein tower: $M = l_1/R_1$ where l_1 can take any integer value. Returning to the case of finite-n we have instead,

$$M = \frac{n}{\pi R_1} \sin\left(\frac{l_1 \pi}{n}\right). \tag{4.8}$$

Now the quantized momentum, l_1, is only defined modulo n and therefore naturally takes values in $\mathbb{Z}_n$ rather than $\mathbb{Z}$. This 'compactification' of the momentum space is precisely what happens when a continuous spatial dimension is discretized. In fact the full mass formula (4.4) is exactly the one you would get by replacing the continuous torus in spacetime with an $n \times n$ lattice with spacings,

$$\varepsilon_1 = \frac{2\pi R_1}{n} = \frac{1}{|\alpha_1|} \qquad \varepsilon_2 = \frac{2\pi R_2}{n} = \frac{1}{|\alpha_2|} \tag{4.9}$$

and replacing the kinetic terms with appropriate finite differences.

In summary the full spectrum of the β-deformed theory on the Higgs branch $\mathcal{H}_m$ coincides with that of a six-dimensional U(m) gauge theory with two compact discrete dimensions and four non-compact continuous dimensions. While the lattice theory preserves $\mathcal{N} = 1$ supersymmetry, the full classical spectrum of toroidally compactified six-dimensional $\mathcal{N} = (1,1)$ supersymmetric U(m) Yang-Mills theory is recovered in a 'continuum' limit, $n \to \infty$, $\varepsilon_i \to 0$. The fact that the spectra of the two theories coincide suggests the theories themselves may actually be the same. To make this convincing we must also compare the interactions of the two theories and, in particular, understand how six-dimensional gauge invariance can emerge from our four-dimensional theory. This is the main topic of the next subsection.

4.2 The lattice action

In our chosen vacua, the scalar vacuum expectation values can be written as $\langle \Phi_i \rangle = \alpha_i \Gamma_i$ for $i = 1, 2$ with,

$$\Gamma_1 = I_{(m)} \otimes U_{(n)} \qquad \Gamma_2 = I_{(m)} \otimes V_{(n)} \tag{4.10}$$

and $\langle \Phi_3 \rangle = 0$.

Following [10], we expand the scalar fields Φ_1 and Φ_2 around their vacuum values as,

$$\Phi_i = \left(I_{(N)} + \varepsilon_i H_i\right) U_i \langle \Phi_i \rangle \tag{4.11}$$

for $i = 1, 2$. The fluctuations are parametrized in terms of two $N \times N$ hermitean matrices H_1, H_2 and two $N \times N$ unitary matrices U_1 and U_2. The (spontaneously-broken) U(N) gauge symmetry of the β-deformed theory acts on these fields as,

$$H_i \to G H_i G^\dagger \qquad U_i \Gamma_i \to G U_i \Gamma_i G^\dagger \tag{4.12}$$

where $G \in$ U(N) and $i = 1, 2$.

It is instructive to focus on the action of the unitary fluctuations U_i, setting the other fluctuating fields to zero. On substituting (4.11) for Φ_1 and Φ_2 in the scalar potential of the β-deformed theory we obtain,

$$V = \frac{|\alpha_1|^2|\alpha_2|^2}{4g^2} \sum_{i\neq j} \mathrm{Tr}_N \left[U_i \left(\Gamma_i U_j \Gamma_i^\dagger \right) \left(\Gamma_j U_i^\dagger \Gamma_j^\dagger \right) U_j^\dagger \right] . \tag{4.13}$$

The zero-dimensional matrix model with action (4.13) is known as the twisted Eguchi-Kawai model and its equivalence to a *non-commutative* lattice gauge theory is well known. We will now review this connection following the derivation of this result given in [33] (see in particular section 7.2 of this reference).

To understand how a two-dimensional lattice emerges we will consider the expansion (4.3) of an arbitrary $N \times N$ matrix field $A(y)$ in terms of the basis elements $J^{(\vec{l})}$ with $m \times m$ coefficients $a^{(\vec{l})}(y)$. Here and in the following, y_μ denotes the coordinates of the four-dimensional spacetime. Our explicit calculation of the spectrum described above indicates that the $m \times m$ matrix fields $a^{(\vec{l})}(y)$ are mass eigenstates with eigenvalue $M^{(\vec{l})}$. In our Kaluza-Klein interpretation of the spectrum described, the vector $\vec{l} \in \mathbb{Z}_n^2$ is identified with the discrete momentum on the two dimensional torus. Hence the modes $a^{(\vec{l})}$ should be thought of as the Fourier modes of a six dimensional lattice field.

We can make this correspondence explicit by defining a periodic $n \times n$ lattice $\mathcal{L}$ with points,

$$\vec{x} = (x_1, x_2) = (n_1\varepsilon_1, n_2\varepsilon_2) \in \mathcal{L} \tag{4.14}$$

where the integers n_1 and n_2 are defined modulo n. The lattice spacings ε_1 and ε_2 are defined in (4.9) above. For any $N \times N$ matrix field in four dimensions $A(y)$ we then introduce a corresponding $m \times m$ lattice field $\mathcal{A}(\vec{x}, y)$ defined by,

$$\begin{aligned} \mathcal{A}(\vec{x}, y) &= \frac{1}{n^2} \sum_{\vec{l}\in\mathbb{Z}_n^2} a^{(\vec{l})}(y) \exp\left(i\frac{l_1 x_1}{R_1} + i\frac{l_2 x_2}{R_2} \right) \\ &= \frac{1}{n^2} \sum_{\vec{l}\in\mathbb{Z}_n^2} a^{(\vec{l})}(y) \exp\left(\frac{2\pi i}{n} \vec{l} \cdot \vec{n} \right) \end{aligned} \tag{4.15}$$

where the coefficients $a^{(\vec{l})}(y)$ are defined as in (4.3) and $\vec{n} = (n_1, n_2) \in \mathbb{Z}_n^2$. By standard standard properties of the basis elements $J^{(\vec{l})}$ defined in (4.2), this defines a one-to-one invertible map Ω between $N \times N$ matrix fields in four dimensions and $m \times m$ matrix fields defined on the product of four-dimensional spacetime with the lattice $\mathcal{L}$ defined above.

$$\Omega \ : \quad A(y) \ \leftrightarrow \ \Omega[A] = \mathcal{A}(\vec{x}, y) . \tag{4.16}$$

The map is a finite dimensional version of the Wigner-Weyl correspondence between classical fields and operators. For simplicity we will suppress dependence on the four-dimensional coordinate y_μ in the following.

JHEP08(2004)043

Applying the map Ω to the unitary fluctuation matrices U_1 and U_2 yields two $m \times m$ lattice fields: $\mathcal{U}_i(\vec{x}) = \Omega[U_i]$ for $i = 1, 2$. Thus we repackage the $n^2 m^2$ degrees of freedom in each $N \times N$ matrix U_i as an $m \times m$ matrix field $\mathcal{U}_i(\vec{x})$ defined on an $n \times n$ lattice. The significance of this becomes clear when we rewrite the Eguchi-Kawai action (4.13) in terms of the lattice fields. The result is,

$$V = \frac{|\alpha_1|^2 |\alpha_2|^2}{4g^2 n} \sum_{i \neq j} \sum_{\vec{x} \in \mathcal{L}} \mathrm{Tr}_m \left[\mathcal{U}_i(\vec{x}) \star \mathcal{U}_j(\vec{x} + \epsilon_i \hat{\imath}) \star \mathcal{U}_i^\dagger(\vec{x} + \epsilon_j \hat{\jmath}) \star \mathcal{U}_j^\dagger(\vec{x}) \right] \tag{4.17}$$

where $\hat{\imath}$ denotes a unit vector in the i'th direction for $i = 1, 2$. We have also introduced a star-product of lattice fields which is the image of $N \times N$ matrix multiplication under the map Ω defined above. For any two lattice fields $\mathcal{A}(\vec{x}) = \Omega[A]$ and $\mathcal{B}(\vec{x}) = \Omega[B]$ we define $\mathcal{A} \star \mathcal{B} = \Omega[AB]$. Thus the unitary matrices U_i are mapped to *star-unitary* lattice fields satisfying $\mathcal{U}_i(\vec{x}) \star \mathcal{U}_i^\dagger(\vec{x}) = \mathcal{U}_i^\dagger(\vec{x}) \star \mathcal{U}_i(\vec{x}) = I_{(m)}$. As we now explain, (4.17) is the action of a non-commutative lattice gauge theory.

The explicit formula for the lattice star product is,

$$\mathcal{A}(\vec{x}) \star \mathcal{B}(\vec{x}) = \frac{1}{n^2} \sum_{\vec{y}, \vec{z} \in \mathcal{L}} \mathcal{A}(\vec{x} + \vec{y}) \mathcal{B}(\vec{x} + \vec{z}) \, \exp\left(2i \, \vec{y} \cdot \vartheta^{-1} \cdot \vec{z}\right) \tag{4.18}$$

where the two-form $\vartheta_{ij} = \vartheta \varepsilon_{ij}$ with,

$$\vartheta = \frac{n}{2\pi} \varepsilon_1 \varepsilon_2 \,. \tag{4.19}$$

The lattice star-product is a discretized form of the Groenewald-Moyal (GM) star product used to define quantum field theory on a continuum non-commutative space. In particular, the lattice sums appearing in (4.18) are finite-dimensional approximations to the space-time integrals appearing in the standard definition of the continuum GM star-product. Like its continuum counterpart, the lattice star-product is associative but non-commutative even in the abelian case $m = 1$. The length-scale at which non-commutativity becomes significant is set by the parameter $\sqrt{\vartheta}$. Taking a formal limit in which $\vartheta \to 0$ the lattice star product becomes an ordinary product of $m \times m$ matrices. In this limit, the lattice action (4.17) becomes precisely the standard Wilson placquette action for a $\mathrm{U}(m)$ lattice gauge theory. The action (4.17) can therefore be thought of as a non-commutative deformation of $\mathrm{U}(m)$ lattice gauge theory and it was proposed in [34] (see also [35]) as a proper non-perturbative definition of the corresponding continuum non-commutative gauge theory.

For non-zero ϑ, the $\mathrm{U}(m)$ gauge symmetry of the Wilson action is deformed to a star-gauge symmetry. In the present context this symmetry is actually a direct consequence of the $\mathrm{U}(N)$ gauge symmetry of the β-deformed theory. For each element $G \in \mathrm{U}(N)$, we introduce an $m \times m$ matrix field $\mathcal{G}(\vec{x}) = \Omega[G]$ defined at each lattice point. As G is unitary, its image $\mathcal{G}$ under Ω is constrained to be star-unitary: $\mathcal{G}(\vec{x}) \star \mathcal{G}^\dagger(\vec{x}) = \mathcal{G}^\dagger(\vec{x}) \star \mathcal{G}(\vec{x}) = I_{(m)}$. The invariance of the β-deformed theory under the $\mathrm{U}(N)$ gauge transformation (4.12) then implies that the non-commutative lattice action (4.17) is then invariant under the star gauge transformation;

$$\mathcal{U}_i(\vec{x}) \ \to \ \mathcal{G}(\vec{x}) \star \mathcal{U}_i(\vec{x}) \star \mathcal{G}^\dagger(\vec{x} + \varepsilon_i \hat{\imath}) \tag{4.20}$$

For $\vartheta \to 0$ this reduces to the standard $\mathrm{U}(m)$ gauge invariance of the Wilson lattice action.

JHEP08(2004)043

So far we have only considered the unitary fluctuations U_1 and U_2 of the scalar fields Φ_1 and Φ_2 which give rise to lattice gauge fields $\mathcal{U}_i(\vec{x})$. Among the other fields we need to restore are the hermitean fluctuations $H_1(y)$ and $H_2(y)$ defined in (4.11). Under the map Ω, these yield $m \times m$ lattice fields $\mathcal{H}_i = \Omega[H_i]$ for $i = 1, 2$. As in (4.12), the fields H_i transform in the adjoint representation of the U(N) gauge group. This means that the resulting lattice fields $\mathcal{H}_i$ transform in the adjoint of the U(m) star-gauge transformation,

$$\mathcal{H}(\vec{x}) \;\rightarrow\; \mathcal{G}(\vec{x}) \star \mathcal{H}(\vec{x}) \star \mathcal{G}^{\dagger}(\vec{x}) \tag{4.21}$$

for $i = 1, 2$. Note the difference between the transformation properties of H_i and U_i in (4.12) which leads to the different transformations (4.21) and (4.20) for the corresponding lattice fields. Thus the fields $\mathcal{H}_i$ transform like adjoint-valued scalar fields in the resulting six-dimensional theory, while the $\mathcal{U}_i$ correspond to the components of a six-dimensional gauge field in the two discrete directions.

By applying the map Ω to each field of the β-deformed theory, it is possible to rewrite the whole action of the four-dimensional theory on the Higgs branch $\mathcal{H}_m$ as a six-dimensional non-commutative U(m) gauge theory with two compact discrete dimensions [10]. The resulting action is invariant under U(m) star-gauge transformations and under Lorentz transformations of the four continuous dimensions. It also invariant under the action of four supercharges inherited from the β-deformed theory. In fact the resulting theory is a discretized version of six-dimensional non-commutative $\mathcal{N} = (1, 1)$ supersymmetric Yang-Mills theory with gauge group U(m). This is consistent with the calculation of the spectrum reported in the previous subsection. We will not give the full lattice action here, briefly review the origin of the six-dimensional fields and their interactions.

Under Ω, the four-dimensional U(N) gauge field yields the components of a U(m) gauge field along the four continuous dimensions. Together with the lattice gauge fields $\mathcal{U}_i$, $i = 1, 2$, they define the six dimensional U(m) gauge field. The remaining complex scalar field Φ_3 yields a complex adjoint scalar in six dimensions. Together with the fields $\mathcal{H}_i$ $i = 1, 2$ we have a total of four real adjoint scalars. The fermionic content of the theory includes four species of complex adjoint fermions which transform as Weyl fermions under Lorentz transformations of the four continuous dimensions. This is precisely the matter content of a U(m) vector multiplet of $\mathcal{N} = (1, 1)$ supersymmetry in six dimensions. Each of these fields has a discretized kinetic term and the spacetime derivatives appearing are covariant with respect to star-gauge transformations. The action also includes a commutator potential for the adjoint scalars which matches that of the $\mathcal{N} = (1, 1)$ theory in six dimensions. The interactions of the fermionic fields are determined by the four unbroken supersymmetries of the lattice action.

The non-commutative lattice action (4.17) has a standard classical continuum limit obtained by taking an $n \to \infty$ limit in which the lattice spacings ε_i go to zero while holding the length-scale of non-commutativity $\sqrt{\vartheta} = \sqrt{n\varepsilon_1\varepsilon_2/2\pi}$ fixed. The resulting scaling $\varepsilon_i \sim 1/\sqrt{n}$ for $i = 1, 2$, mean that the radii of the torus $R_i = n\varepsilon_i/2\pi$ go to infinity like $\sqrt{n}$. In this limit the lattice action (4.17) goes over to the Maxwell action of a U(m) gauge theory on an infinite non-commutative plane R^2_{ϑ}. Restoring the extra four commutative dimensions of the β-deformed theory and the additional field content described above we recover a maximally supersymmetric six-dimensional U(m) gauge theory on $R^{3,1} \times R^2_\vartheta$.

As explained in [10], we may also understand this result by making contact with the ideas of Matrix theory [36]. As before, we expand the fields of the U(N) β-deformed theory around their background values $\langle \Phi_i \rangle = \alpha_i \Gamma_i$ and take the $n \to \infty$ limit. As the deformation parameter $\beta = 2\pi/n$ goes to zero in this limit we can simply perform the calculation using the undeformed $\mathcal{N} = 4$ lagrangian. For $i = 1, 2$, we formally introduce new fields,

$$\Phi_i = \exp\left(i\varepsilon_i \hat{C}_i\right) \simeq I_{(N)} + i\varepsilon_i \hat{C}_i + \; O(\varepsilon^2) \tag{4.22}$$

where the second equality holds near the continuum limit $n \to \infty$, $\varepsilon_i \sim 1/\sqrt{n} \to 0$. Using the second equality (and performing a suitable rescaling) we can effectively replace Φ_i by $\hat{X}_i = \vartheta_{ij}\hat{C}_j$ in the $\mathcal{N} = 4$ action where the non-commutativity tensor ϑ_{ij} is defined above. The matrices Γ_i appearing in the vacuum values of the Φ_i satisfy $\Gamma_1\Gamma_2 = \bar{\omega}_n \Gamma_2\Gamma_1$. As a consequence, in the vacuum state, the fields $\hat{X}_i$ satisfy,

$$[\hat{X}_i, \hat{X}_j] = i\vartheta_{ij} \,. \tag{4.23}$$

This relation has no solutions for finite $N = mn$, and the required background only exists for $N = \infty$. Expanding the fields of a maximally supersymmetric U(∞) gauge theory in p spacetime dimensions, around this background yields a maximally supersymmetric, non-commutative U(m) gauge theory in $p + 2$ dimensions [37, 38]. Indeed, this is just the standard Matrix theory construction of the worldvolume theory on m D(p+2) branes starting from the theory on $N = mn$ Dp branes.

Another interesting connection discussed in [10] is to the original notion of deconstruction introduced in [16, 17]. In particular, a four-dimensional gauge theory which deconstructs the *commutative* version of the six-dimensional $\mathcal{N} = (1, 1)$ theory considered here was given in [17]. To deconstruct the six-dimensional theory with gauge group U(m) one needs a four-dimensional $\mathcal{N} = 1$ quiver theory with gauge group U$(m)^n$ and certain bifundamental chiral multiplets. From a field theoretic perspective there is no obvious connection between this theory and the β-deformed theory with gauge group U(N).

As explained in [10], the connection only becomes clear when one considers the string theory realisation of the two theories. The quiver theory of [17] is the world volume theory of m D3 branes placed at a $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ singularity. As we mentioned in section 3, the β-deformed theory on $\mathcal{H}_m$ lives on the world-volume of m D3 branes at the same singularity but now with a single unit of discrete torsion. A string-theoretic argument given in [17] explains why the orbifold theory naturally gives rise to a six-dimensional lattice gauge theory. One of the main points of [10] is that the same argument goes through in the presence of discrete torsion. The sole modification caused by the introduction of discrete torsion is that the resulting lattice gauge theory becomes non-commutative.

Finally we could also repeat the analysis of this section in a more general Higgs branch vacuum with $\mathcal{H}_m$ with $\alpha_3 \neq 0$. It is claimed in [10] that this leads to a non-rectangular or slanted torus. We have not checked this directly, but it will be confirmed in the string theory construction of section 8.

JHEP08(2004)043

5. The low-energy effective theory

5.1 The classical theory

In the previous section, we have argued that, for $N = nm$, the U(N) β-deformed theory in the Higgs branch vacuum (3.16) specified by complex numbers α_1 and α_2 is classically equivalent to a certain six-dimensional theory. The six-dimensional theory is a U(m) gauge theory compactified on an $n \times n$ lattice with periodic boundary conditions. Collecting various formulae given above, the parameters of the six dimensional theory can be expressed in terms of the four-dimensional parameters. The lattice spacings of the two compact discrete dimensions are,

$$\varepsilon_1 = \frac{1}{|\alpha_1|} \qquad \varepsilon_2 = \frac{1}{|\alpha_2|} \tag{5.1}$$

The radii of the two-dimensional torus which they define are,

$$R_1 = \frac{n}{2\pi|\alpha_1|} \qquad R_2 = \frac{n}{2\pi|\alpha_2|} \tag{5.2}$$

The six-dimensional gauge coupling and non-commutivity parameter are given by,

$$G_6 = \frac{g^2 n}{|\alpha_1||\alpha_2|} \qquad \vartheta = \frac{n}{2\pi|\alpha_1||\alpha_2|} \tag{5.3}$$

respectively. As is conventional, we will also define a dimensionless non-commutativity parameter $\Theta = \vartheta/2\pi R_1 R_2 = 1/n$.

Importantly our derivation of the six dimensional action was based on a classical analysis of the β-deformed theory. This is only valid when the four dimensional t' Hooft couping $\lambda = g^2 N$ is much less than one. In this regime the dynamical-length scale of the six-dimensional gauge theory set by the gauge coupling G_6 is much smaller than the lattice spacing ε. Thus the six-dimensional lattice gauge theory is far from its continuum limit. However, even at weak coupling, we can obtain an effective continuum theory by focussing on modes of wavelength much larger than the lattice spacing. Writing a classical effective action for these modes we can replace discrete derivatives by continuous ones and obtain a continuum effective action. This proceedure is identical to taking the classical continuum limit discussed above. Hence our long-wavelength effective action, valid on length scales much larger than $\max\{\varepsilon_1, \varepsilon_2\}$, is $\mathcal{N} = (1,1)$ supersymmetric U(m) gauge theory on $R^{3,1} \times T^2_\Theta$.

5.2 Morita equivalence

As the the effective theory is defined on a non-commutative torus with rational non-commutativity parameter, $\Theta = 1/n$, it can be related by Morita duality to a commutative theory (See [33] for a review of this topic). In the present case, the Morita dual theory is a six dimensional $\mathcal{N} = (1,1)$ gauge theory with gauge group $G = $ U(N) defined on the commutative space $T^2 \times R^{3,1}$. The commutative torus in question is rectangular and has sides has sides of length;

$$R'_1 = \frac{1}{n} R_1 = \frac{1}{2\pi|\alpha_1|} \qquad R'_2 = \frac{1}{n} R_2 = \frac{1}{2\pi|\alpha_2|}\,. \tag{5.4}$$

The dual six-dimensional gauge coupling is,

$$G_6'^2 = \frac{G_6^2}{n} = \frac{g^2}{|\alpha_1||\alpha_2|}\,. \tag{5.5}$$

The theory has non-trivial boundary conditions on T^2. If we choose coordinates $\vec{x} = (x_1, x_2)$ on T^2 with $0 \leq x_1 < 2\pi R_1'$ and $0 \leq x_2 < 2\pi R_2'$, then all adjoint fields obey the 't Hooft boundary conditions;

$$\begin{aligned} A'(x_1 + 2\pi R_1', x_2) &= \Gamma_1 A'(x_1, x_2)\Gamma_1^\dagger \\ A'(x_1, x_2 + 2\pi R_2') &= \Gamma_2 A'(x_1, x_2)\Gamma_2^\dagger \end{aligned} \tag{5.6}$$

where the matrices Γ_1 and Γ_2 are defined in (4.10) above.

The equivalence is a one-to-one map between classical field configurations in the two theories. We start from the non-commutative U(m) effective theory, choosing coordinates $\hat{\vec{x}} = (\hat{x}_1, \hat{x}_2)$ on the torus T^2_Θ with $0 \leq \hat{x}_1 < 2\pi R_1$ and $0 \leq \hat{x}_2 < 2\pi R_2$. Every single valued adjoint field on T^2_Θ has a Fourier expansion of the form,

$$\mathcal{A}(\vec{x}, y) = \frac{1}{n^2}\sum_{\vec{l}\in\mathbb{Z}^2} a^{(\vec{l})}(y)\, \exp\left(2\pi i\left[\frac{l_1\hat{x}_1}{R_1} + \frac{l_2\hat{x}_2}{R_2}\right]\right) \tag{5.7}$$

where, as above, y_μ denotes the coordinates of the four commutative directions. This expansion is just the continuum version of the expansion (4.15) of the corresponding field defined on the lattice $\mathcal{L}$. In particular, the coefficients $a^{(\vec{l})}$, with $\vec{l} \in \mathbb{Z}^2$, appearing in (5.7) are the large-n limit of the corresponding coefficients in (4.15) where $\vec{l} \in \mathbb{Z}^2_n$.

A choice of the $m \times m$ Fourier modes $a^{\vec{l}}(y)$ of each adjoint-valued field uniquely defines a field configuration in our U(m) effective theory. We can define a corresponding field configuration in the dual U(N) theory, by using the same Fourier coefficients. Specifically each adjoint-valued field of the dual theory is an $N \times N$ matrix which we write as,

$$\mathcal{A}'(\vec{x}, y) = \frac{1}{n^2}\sum_{\vec{l}\in\mathbb{Z}^2} a^{(\vec{l})}(y)\, \otimes\, V_{(n)}^{l_1} U_{(n)}^{-l_2}\omega_n^{\frac{l_1 l_2}{2}}\, \exp\left(2\pi i\left[\frac{l_1 x_1}{nR_1'} + \frac{l_2 x_2}{nR_2'}\right]\right) \tag{5.8}$$

It is straightforward to check that this field configuration satisfies the approriate 't Hooft boundary conditions (5.6) on the dual commutative torus.

The precise statement of classical Morita equivalence is that the Yang-Mills action of the configurations $\mathcal{A}$ and $\mathcal{A}'$ is the same. As will be important in the following, various topological charges carried by the two dual field configurations are mapped onto each other [39]. As in lower dimensions the equivalence between the two theories can be thought of as a formal change of variables in the path integral. In dimension less than or equal to four this implies an exact equivalence between the theories at the quantum level which can actually be realised graph by graph in the Feynman diagram expansion. In the present case, both continuum theories are non-renormalisable and so the status of Morita equivalence in the quantum theory is less clear. In the following we, will only use the equivalence as a map between classical configurations.

For the present purposes, the key point is that the dual six-dimensional U(N) formulation of the low-energy effective theory can actually be related to the original four-dimensional U(N) gauge theory we started with. To make contact with the four dimensional theory we simply dimensionally reduce the six-dimensional dual gauge theory to a four dimensional one by setting $\vec{x} = \vec{0}$ in (5.8). Comparing (5.7) with (4.15) and (5.8) with (4.3), we see that the relation between the six dimensional U(m) adjoint field $A(\vec{x}, y)$ and the four-dimensional U(N) adjoint field $A'(\vec{0}, y)$ is simply the large-n limit of the the one-to-one map Ω between matrices and lattice fields introduced in the previous section. The upshot of this is that, at large-n, we have an explicit map between the six-dimensional U(m) fields of the effective theory and the U(N) fields of the four-dimensional β-deformed theory we started with. For example, if we take the components $\mathcal{A}_\mu(\vec{x}, y)$ for $\mu = 0, 1, 2, 3$ of the U(m) gauge field in the four commutative directions, we can actually identify the dual U(N) field $\mathcal{A}'_\mu(\vec{0}, y)$ with the U(N) gauge field of the four-dimensional β-deformed gauge theory. In the next section, we will use this correspondence to interpret the topological charges of the U(m) effective theory in terms of those of the four-dimensional theory.

5.3 Quantum corrections

The classical derivation of the effective theory presented above is only valid at weak 't Hooft coupling: $g^2N \ll 1$. In this subsection , we will consider how the classical picture is modified by quantum corrections. As explained in section 3, the Higgs branch $\mathcal{H}_m$ cannot be lifted because of the $\mathcal{N} = 1$ superconformal invariance and the other non-anomalous global symmetries. Thus the massless spectrum will remain the same in the quantum theory. In section 3, we argued that the complex structure of the Higgs branch is also protected from quantum corrections by the symmetries of the theory.

On the other hand, it is obvious that the low-energy effective theory with parameters as given in (5.1), (5.2), (5.3) must be modified in the quantum theory. Gauge theories in six dimensions become strongly-coupled in the UV. In particular, we expect that perturbation theory breaks down on length-scales smaller than the scale, $\sqrt{G_6^2 m}$, set by the six-dimensional 't Hooft coupling. Thus the low-energy theory described above only makes sense when this scale is much smaller than the lattice spacing ε. From (5.1) and (5.3), we see that this is only true when $g^2N \ll 1$. Away from weak coupling, the situation is less clear. If the lattice spacing ε is really smaller than the lengthscale set by the 't Hooft coupling, then the effective theory cannot simply be a six dimensional gauge theory: some new physics must be included on length-scales of order $\sqrt{G_6^2 m} = \sqrt{g^2N/|\alpha_1||\alpha_2|}$. We will start to address this issue in sections 8 and 9 and will consider it further in [1].

The theory on the Higgs branch $\mathcal{H}_m$, exhibits a key simplification in the large-N limit which we have already touched on above. As the deformation parameter $\beta = 2\pi/n$, with $n = N/m$, the theory recovers $\mathcal{N} = 4$ supersymmetry in a large-N limit with m and g^2 fixed. Note that the additional supersymmetry appears at the level of the microscopic action and is therefore valid on all length-scales and for any fixed value of the gauge coupling. In this limit the four-dimensional 't Hooft coupling becomes large, $g^2N \gg 1$ and perturbation theory breaks down. Instead the UV theory is well described by IIB supergravity (provided $g^2 \ll 1$). In addition to taking a large-N limit, we are simultaneously taking the deformation parameter $\beta = 2\pi m/N$ to zero. The fact that the β-deformation has a smooth supergravity description suggests that this combined limit exists and makes sense.

The spectrum discussed in section 4.1, consists entirely of BPS-saturated multiplets of the enlarged $\mathcal{N} = 4$ supersymmetry. Their masses are determined by the central charges of the $\mathcal{N} = 4$ supersymmetry algebra which correspond to momenta in the two compact dimensions. In theories with $\mathcal{N} = 4$ supersymmetry, central charges do not recieve quantum corrections. For this reason, we expect that the classical mass formula (4.4) becomes exact in the $N \to \infty$ limit described above. As the enlarged supersymmetry is valid for all length scales, this should be true even for states near the edge of the Brillouin zone with momenta $l_i \sim n = N/m$. The enlarged supersymmetry also forces the effective action for the massless degrees of freedom to be $\mathcal{N} = 4$ SUSY Yang-Mills with gauge group U(m) as it was in the classical case. The low energy gauge coupling is exactly given by the replacement $\tau \to \tau_R$ in the classical formula $\tau_H = N\tau$.

To summarize our conclusions, the Higgs branch $\mathcal{H}_m$ exists and has the same massless spectrum for all values of the parameters. Away from the semiclassical regime $g^2 N \ll 1$, the effective theory cannot simply be the six-dimensional gauge theory which appears classically. However, in the $N \to \infty$ limit with fixed g^2, the spectrum of Kaluza-Klein modes and the effective action for the massless degrees of freedom are exactly the same as they are in the classical theory.

6. Instanton strings and magnetic confinement

In this section we will focus on the classical solutions of the U(m) effective theory. An important fact about non-abelian gauge theories in (5+1)-dimensions is that they possess exact classical solutions corresponding to finite tension strings. The solutions are obtained by embedding the familiar Yang-Mills instantons in four of the five spatial dimensions of the effective theory. The resulting object is then interpreted as a string infinitely stretched in the remaining spatial dimension. The string tension can be deduced from the action of a 4d instanton;

$$T = \frac{8\pi^2}{G_6^2} = \frac{8\pi^2|\alpha_1||\alpha_2|}{g^2 n}\,. \tag{6.1}$$

The string tension T saturates a Bogomol'nyi lower bound and the corresponding self-dual gauge field configuration preserves eight of the sixteen supercharges of the $\mathcal{N} = (1,1)$ theory. We will refer to these objects as BPS strings in the following. It is important to bear in mind that these strings are BPS solutions only of the low-energy effective theory. At finite N, they are certainly not BPS configurations of the full theory which only has four supercharges and has no central charges corresponding to strings or particles. As $N \to \infty$ the sixteen unbroken supersymmetries are recovered. Similar arguments to those given in the previous section suggest that the semiclassical formula (6.1) for the tension becomes exact in this limit.

The classical effective action introduced above can safely be used to describe field configurations whose typical spatial variation in the two discretized dimensions has wavelength much larger than the lattice spacing. The gauge fields of ordinary non-abelian

Yang-Mills instanton are characterised by a variable scale size ρ and they will be correctly described by the continuum effective action if this size is much larger than the lattice spacing; $\rho \gg \max\{\varepsilon_1, \varepsilon_2\}$.

As the effective theory is formulated on $T^2_\Theta \times R^{3,1}$ (or $T^2_\Theta \times R^4$ in the euclidean case), the Yang-Mills instanton gives rise to three basic types of configuration:

Long Strings. In this case we consider a static string of infinite length stretched in one of the three spatial directions of $R^{3,1}$. The classical field configuration is equivalent to a single Yang-Mills instanton of a four-dimensional euclidean U(m) gauge theory on the non-commutative space $T^2_\Theta \times R^2$ with $\Theta = 1/n$.

Winding Modes. Here we consider a static string wound q_1, q_2 times around the x_1, x_2 dimensions of T^2_Θ. This gives rise to a towers of BPS solitons with masses,

$$M^2_{(q_1,q_2)} = T^2\left(q_1^2 R_1^2 + q_2^2 R_2^2\right) = \left(\frac{8\pi^2}{g^2}\right)^2 \left(q_1^2|\alpha_2|^2 + q_2^2|\alpha_1|^2\right). \tag{6.2}$$

Configurations with $q_2 = 0$ correspond to q_1 self-dual U(m) Yang-Mills instantons on $R^3 \times S^1$ where the radius of the compact dimension is $R_1 = n/2\pi|\alpha_1|$.

Worldsheet Instantons. Finally, in the euclidean theory, we can consider the case of a euclidean string worldsheet wrapped p times around the torus T^2_Θ. This yields an instanton of action,

$$S = pT\,4\pi^2 R_1 R_2 = \frac{8\pi^2 np}{g^2}. \tag{6.3}$$

Each of the objects listed above can be realised as a classical solution of the six-dimensional U(m) effective theory. As discussed above, the description will be valid provided the scale-size of the corresponding instanton is much larger than the lattice spacing. The formulae of the previous section allow us to map these back to configurations of the original U(N) fields of the β-deformed theory. In the rest of this section we will see that the long strings discussed above actually correspond to stable classical solutions of the full β-deformed theory, labelled by a conserved topological charge. More generally, some of the other configurations discussed above may not correspond solutions of the full theory and may have instabilities. In this case, the prediction of deconstruction is that these objects become stable in the large-n limit. In the quantum theory at finite n, these will correspond to unstable states with typical life time τ proportional to a positive power of n.

In a Higgs phase vacuum, we typically expect to find the confinement of magnetic charges. The usual signal of this confinement is the formation of magnetic flux tubes of finite tension which can end on magnetic charges. Now we have seen that the low energy effective theory on $\mathcal{H}_m$ does indeed possess string-like solutions and it is natural to guess that these strings are precisely the expected flux tubes. In the following we will confirm this guess using both field theory and string theory arguments.

The relation between the Higgs mechanism and magnetic confinement is clearest when the theory has a mass gap. In this case 't Hooft's standard classification of possible massive phases is applicable [41]. However, for the β-deformed theory, there can never be a mass gap. In addition to the massless photon of the unbroken central U(1) subgroup of U(N), the Higgs-phase theory always has massless scalars due to the spontaneous breaking of conformal invariance. In theories with massless scalars, the existence of stable magnetic flux tubes is not guarenteed. Recent work by several authors [42] has illustrated several subtleties of the massless case and thrown doubt onto the occurence of magnetic confinement in these theories. In the present case we will see that, despite the presence of massless scalars, stable magnetic flux tubes do emerge in the large-n limit.

In any non-abelian gauge theory the possible magnetic charges and fluxes are classified by the first homotopy class of the gauge group. In the present case we have $\pi_1(\mathrm{U}(N)) = \mathbb{Z}$. The non-abelian magnetic flux is therefore specified by the choice of an integer k. In a theory with all fields in the adjoint representation, the photon of the central U(1) subgroup of U(N) decouples.[9] Hence we decompose U(N) in the standard way as,

$$\mathrm{U}(N) \equiv \frac{\mathrm{U}(1) \times \mathrm{SU}(N)}{\mathbb{Z}_N}\,. \tag{6.4}$$

Under this decomposition, we can identify a U(1) magnetic flux, $l \in \pi_1[\mathrm{U}(1)] = \mathbb{Z}$ (which is equal to the total U(N) magnetic flux k) and the so-called 't Hooft non-abelian magnetic flux $[m] \in \pi_1[\mathrm{SU}(N)/\mathbb{Z}_N] = \mathbb{Z}_N$. The decomposition (6.4) implies the selection rule $m = -l = -k \bmod N$ which correlates the 't Hooft flux with the total U(N) flux. A flux tube carrying k units of flux can end on a magnetic charge of quantized value $\pm k$ The formation of stable finite-tension strings carrying integer multiples of magnetic flux k leads to the linear confinement of objects carrying magnetic charges which are integer multiples of $\pm k$.

In the light of the above discussion we would like to show that the instanton string solutions of the six-dimensional low-energy effective theory actually correspond to configurations carrying U(N) magnetic flux in the original β-deformed theory. This follows from the standard properties of the Morita equivalence discussed in the previous section. In particular we start from a long U(m) instanton string, stretched in the y_3 direction. The string corresponds to a static classical configuration of the six-dimensional U(m) gauge connection[10] $\mathcal{A}(\vec{x}, y)$. which depends only on the four spatial coordinates $(\hat{x}_1, \hat{x}_2, y_1, y_2)$ which parameterize a submanifold $T^2_\Theta \times R^2_{12}$ of the full six-dimensional spacetime $T^2_\Theta \times R^{3,1}$. The instanton string is characterized by a non-zero second Chern class,

$$k = \frac{1}{8\pi^2}\int_{T^2_\Theta \times R^2_{12}} \mathrm{Tr}_m\left[\mathcal{F}\wedge\mathcal{F}\right] \in \mathbb{Z} \tag{6.5}$$

where $\mathcal{F}$ is the corresponding curvature two-form. The Morita equivalence discussed in the previous section maps the non-commutative U(m) connection $\mathcal{A}$ to a commutative U(N) connection $\mathcal{A}'$ defined on $T^2 \times R^{3,1}$ with ' t Hooft boundary conditions on T^2. By construction the resulting field configuration only depends on the four coordinates

[9] We remind the reader that this does not mean that the U(N) and SU(N) theories are related in a simple way. In particular the former theory has three additional neutral chiral multiplets.

[10] In this section we will avoid introducing six-dimensional Lorentz indices and work in terms of differential forms instead

(x_1, x_2, y_1, y_2). A standard property of the Morita map (see for example [39]) is that it maps second Chern class (6.5) in Theory II to a first Chern class of Theory I, defined as

$$k' = \frac{1}{2\pi}\int_{R^2_{12}} \mathrm{Tr}_N\left[\mathcal{F}'\right] \in \mathbb{Z}\,. \tag{6.6}$$

Specifically the resulting gauge field configuration has $k' = k$. As we are integrating over R^2_{12} only and not over T^2, equation (6.6) holds pointwise on T^2. In particular it still holds if we set $x_1 = x_2 = 0$. As discussed in the previous section, in this case we can simply identify the components of $\mathcal{A}'(\vec{0}, y)$ along R^2_{12} with the corresponding components of the original four-dimensional gauge connection a. This means that,

$$\frac{1}{2\pi}\int_{R^2_{12}} \mathrm{Tr}_N\left[f\right] = k \tag{6.7}$$

where f is the curvature of the four-dimensional gauge connection A. In the four-dimensional theory, the first Chern class (6.7) also coincides with the total $\mathrm{U}(N)$ magnetic flux in the $(1, 2)$-plane. Further, as the original $\mathrm{U}(m)$ instanton solution is localised in the transverse $T^2_\Theta \times R^2$, each of the Fourier coefficients of its field strength and energy density is localized in the commutative coordinates y_1 and y_2, near the center of the instanton in R^2_{12}. Thus the resulting configuration in the four dimensional theory corresponds to a magnetic string carrying k units of magnetic flux. In particular a single instanton corresponds to a string carrying the minimal quantum of magnetic flux. The core size of the flux tube is dictated by the scale-size ρ of the corresponding instanton.

A more detailed picture of the resulting flux tubes can be gained by using a few simple facts about Yang-Mills instantons (see for example [43]). To make the discussion as simple as possible, we work on square torus by setting $|\alpha_1| = |\alpha_2| = |\alpha|$. In this case we can identify four relevant length-scales in ascending order,

$L_1 = \sqrt{G_6^2 m} = \sqrt{g^2 N}/|\alpha|$. The length scale set by 't Hooft coupling of the six dimensional gauge theory.

$L_2 = 1/|\alpha|$. The lattice spacing.

$L_3 = \sqrt{\vartheta} = \sqrt{2\pi n}/|\alpha|$. The length-scale at which non-commutativity becomes significant.

$L_4 = n/2\pi|\alpha|$. The compactification radius.

First we note that in the large-n semiclassical regime we are considering ($n \gg 1$, $g^2 N \ll 1$) all of these scales are well-separated: $L_1 \ll L_2 \ll L_3 \ll L_4$. As $L_2 \gg L_1$, the six-dimensional theory is weakly coupled on scales larger than the lattice spacing justifying a classical analysis. As noted above, the six-dimensional continuum effective action is valid for describing instantons with scale-size $\rho \gg L_2$. In a non-commutative theory instantons typically have a minimum scale size of order the non-commutativity length scale $\rho_{\rm min} \sim L_3 \gg L_2$. Thus all the low-energy action is valid for all the solutions we will consider. As above, a long string corresponds to a Yang-Mills instanton on the non-commutative space $T^2_\Theta \times R^2_{12}$. Such objects have been studied before and a version of the Nahm transform exists for constructing the corresponding moduli space of solutions [44]. However, for the present purposes a much simpler description will suffice. For $n \gg 1$, there is large heirarchy between the length scale set by the non-commutativity and the radius of compactification. In fact we have $L_4/L_3 \sim \sqrt{n} \to \infty$. We will assume[11] that instantons scale-size $\rho \ll L_4$ exist and are well approximated by instantons on the non-compact euclidean space $R^2_\vartheta \times R^2_{12}$ about which much more is known.

We begin with the case $m = 1$ where the U(N) gauge symmetry is broken down to U(1). In this case we find a non-commutative U(1) instanton with a fixed scale-size $\rho \sim L_3 = \sqrt{2\pi n}/|\alpha|$. As noted above, this length-scale is larger than the lattice spacing by a factor of $\sqrt{n}$ justifying the use of continuum effective field theory. The presence of magnetic flux tubes of fixed size indicates that magnetic charges are confined in the large-n limit. In this limit the central U(1) factor in the gauge group effectively decouples and we can think of these strings as magnetic vortices carrying an N-valued topological charge corresponding to $\pi_1[\mathrm{SU}(N)/\mathbb{Z}_N] = \mathbb{Z}_N$.

Extending the discussion to the Higgs branch $\mathcal{H}_m$ where the unbroken gauge group is U(m), we must consider U(m) instantons on $R^2_\vartheta \times R^2_{12}$. The occurence of a variable scale-size is related to the fact that the gauge group is only partially broken. Magnetic flux in the unbroken U(m) is not confined and should therefore be able to spread out. Increasing the scale size of the instanton corresponds to increasing the core-size of the corresponding magnetic flux tube. We interpret this as the spreading out of the flux in the unbroken part of the gauge group. This interpretation is reinforced by recalling that the U(m) instanton is not invariant under global U(m) gauge rotations. Instead it has a fixed orientation inside the gauge group and the corresponding gauge orbit appears as the coset $\mathrm{U}(m)/\mathrm{U}(m-2)$ in its moduli space.

Finally it is interesting to consider what happens when we move onto the Coulomb branch of the low-energy U(m) theory. In this case the gauge group is further Higgsed down to U$(1)^m$. The effect of introducing such a VEV has on Yang-Mills instanton is well-known. It introduces a potential on the instanton moduli space. In the case of a single instanton we find a potential $V \sim v^2\rho^2$, where v is the scale of the VEV, which tends to force the instanton to zero-size. In the non-commutative case we instead find a potential which looks like $v^2\rho^2$ for large ρ but also has a nontrivial minimum at fixed size $\rho \sim L_3$. Thus we again end up with flux-tubes of fixed core-size and the confinement of the magnetic charge.

In the weak-coupling regime discussed above, the mass-scale $\sqrt{T}$ set by the string tension is large compared to the masses of the elementary quanta of the theory and the strings appear as classical solitons in the low-energy effective action. A semiclassical quantisation of these objects would involve studying a two-dimensional non-linear σ-model with $\mathcal{N} = (4,4)$ supersymmetry with the instanton moduli space as the target. On a branch with unbroken gauge-group U(m) for $m > 1$, the core-size of the string is a modulus of the solution and a semiclassical quantization of the string yields a continuous spectrum of states in six dimensions. On the branches discussed above where the gauge symmetry is further broken to U$(1)^m$, this modulus is lifted and the relevant σ-model now includes

[11]Note that the results of [44] are consistent with this assumption. In the case where Wilson lines are turned off, the resulting moduli space of a single instanton, with fixed COM on non-commutative $R^2 \times T^2$ is the same as in the non-compact case.

a potential term given by the norm of a Killing vector on the moduli space [46]. The potential has isolated minima and we would therefore expect normalizable string winding modes and a discrete spectrum of excited states.

The most interesting questions concerning the strings discussed above relate to their behaviour away from weak coupling. As we have argued above, the semiclassical formula for the string tension $T = 8\pi^2|\alpha_1||\alpha_2|/g^2 n$ becomes exact in the large-N limit. This suggests that there should be interesting regimes in the parameter space where the strings become light and the effective description should be a string theory, rather than a local quantum field theory. In general such a description is only useful if the strings are weakly interacting. In section 9, we will suggest that, at least in one regime of parameters, this is indeed the case.

7. S-duality and electric confinement

The $\mathrm{SL}(2,\mathbb{Z})$ duality transformations (2.13) described in section 2, relate theories with different values of the parameters τ, β and $\kappa = \kappa_{cr}[\tau, \beta]$. In particular they map vacuum states of a theory with one set of values of these parameters onto vacuum states of a theory with another set of values. Vacua on Coulomb or Higgs branches are characterized by non-zero VEVs for the chiral operators $u_{(k_1,k_2,k_3)}$ discussed in section 3. In the $\mathcal{N} = 4$ theory these operators are parts of chiral primary multiplets transforming with fixed modular weights $(k/2, k/2)$ where $k = k_1 + k_2 + k_3$ [25, 26]. As discussed in appendix A, the results of [15] indicate that these operators retain the same weights under the modified $\mathrm{SL}(2,\mathbb{Z})$ transformation (2.13) of the β-deformed theory. Given the VEVs of these operators in a vacuum of one theory, the transformation rule (2.8) specifies their VEVs in a corresponding vacuum state of the dual theory

We now want to apply the S-duality transformation (2.13) given above to the $\mathrm{U}(N)$ theory with $\beta = 2\pi/n$ (where $N = mn$) on its Higgs branch $\mathcal{H}_m$. Specifically we will consider the theory in the vacuum state (3.16) parametrised by three complex numbers α_1, α_2 and α_3. According to Eq (2.13), the dual theory will have deformation parameter $\tilde{\beta} = 2\pi/n(c\tau_R + d)$. The resulting vacuum state can be described in terms of the gauge invariant chiral operators $u_{(k_1,k_2,k_3)}$ defined in section 3. As in the vacuum (3.16), these operators will vanish in the dual vacuum unless *either*

$$(k_1, k_2, k_3) = (0, 0, 0) \bmod n \qquad \text{or} \qquad k_1 = k_2 = k_3 \bmod n\,. \tag{7.1}$$

In the non-vanishing case we have

$$\langle u_{(k_1,k_2,k_3)}\rangle = \exp(i\tilde{\nu}_{(k_1,k_2,k_3)})\tilde{\alpha}_1^{k_1}\tilde{\alpha}_2^{k_2}\tilde{\alpha}_3^{k_3} \tag{7.2}$$

with $\tilde{\alpha}_i = \alpha_i|c\tau_R + d|$ for $i = 1, 2, 3$ where $\exp(i\tilde{\nu}_{(k_1,k_2,k_3)})$ is an unimportant phase.

To simplify our discussion we will focus for a moment on the 'S' generator of $\mathrm{SL}(2,\mathbb{Z})$ and work with $n \gg 1$ so that $\beta = 2\pi/n \ll 1$ and we can work to first order in the perturbation. As explained in section 2, we then have $\kappa_{cr} = 1 + O(1/n^2)$ and which gives

$\tau_R = \tau + O(1/n^2)$. We will also set the vacuum angle $\theta = 0$. In this case the parameters of the theory transform as,

$$g^2 \to \tilde{g}_R^2 = \frac{16\pi^2}{g^2} \qquad \beta = \frac{2\pi}{n} \to \tilde{\beta} = \frac{ig^2}{2n} \tag{7.3}$$

and the dual vacuum has VEVs specified by (7.1), (7.2) with $\tilde{\alpha}_i = \alpha_i(4\pi/g^2)$ for $i = 1, 2, 3$. Several features of this result are worthy of comment;

1. The duality transformation has mapped a theory where the parameter $\beta/2\pi$ takes the rational real value $1/n$ has been mapped to one where this parameter takes an imaginary value $ig^2/4\pi n$. As we start from a three-parameter family of vacua on $\mathcal{H}_m$, the image is a three-parameter family of vacua in the dual theory. In generic vacua, chiral operators such as $u_{(n,0,0)}$, $u_{(0,n,0)}$ and $u_{(0,0,n)}$ all have non-zero VEVs. Our classical analysis showed that branches like these, on which more than one of the three scalar fields Φ_i have a VEV, are only possible for real, rational values of $\beta/2\pi$. Thus the vacua in question lie on a new branch $\tilde{\mathcal{H}}_m$ of the dual quantum theory which is not visible classically.

2. As the branch $\mathcal{H}_m$ exists for all values of the gauge coupling in the theory with $\beta = 2\pi/n$, we deduce that the new branch $\tilde{\mathcal{H}}_m$ exists for all values of the renormalized gauge coupling $\tilde{g}_R^2$ in the β-deformed theory with deformation parameter $\tilde{\beta} = 8\pi^2 i/\tilde{g}_R^2 n$

3. When the original Higgs phase theory is weakly coupled, $g^2 \ll 1/N$, the dual gauge theory has ultra strong coupling $\tilde{g}_R^2 \gg N$. Note however that the converse is not true. When the Higgs theory has very strong coupling $g^2 \gg N$, the dual theory has a very small gauge coupling $\tilde{g}_R^2 \ll 1/N$. However the coefficients appearing in front of the cubic scalar interactions in the superpotential are $\exp(\pm i\beta/2) = \exp(\mp 4\pi^2/\tilde{g}_R^2 n)$. These lead to exponentially large quartic self-interactions of the scalars in this limit. Thus the dual theory is *never* weakly coupled and a semiclassical analysis of its vacuum structure is not reliable. Because of this, the existence of the new branch $\tilde{\mathcal{H}}_m$ does not lead to a contradiction.

4. The β-deformed theory can be thought thought of as a small perturbation of the $\mathcal{N} = 4$ theory provided that $|\beta| = 2\pi/n \ll 1$. In the S-dual vacuum the relevant condition becomes $|\tilde{\beta}| = g^2/2n \ll 1$. Rewriting this in terms of the dual coupling we have $\tilde{g}_R^2 n \simeq \tilde{g}^2 n \gg 1$. Hence, when both $n \gg 1$ and $\tilde{g}^2 n \gg 1$ hold, both theories are close to the $\mathcal{N} = 4$ point. In this regime, the S-duality between the two theories can be established using only the linearized analysis of section 2.

5. The action of the full S-duality group produces new branches in an infinite number of theories with deformation parameter $\tilde{\beta}$ and renormalized gauge couplings $\tilde{\tau}_R$ related as $\tilde{\beta} = 2\pi(c\tilde{\tau}_R - a)/n$. None of the new branches is visible semiclassically.

We can provide further evidence for the existence of the new branches directly from the analysis of chiral condensates in the massive version of the β-deformed theory with

JHEP08(2004)043

gauge group SU(N). This theory has classical superpotential $\mathcal{W}+\Delta\mathcal{W}$ where the new term $\Delta\mathcal{W}$ is given in (2.10). The theory has an SL$(2,\mathbb{Z})$ multiplet of massive vacua in various Higgs and confining phases. The vacua are labelled by three integers r, s and t with $N=rs$ and $t=0,2,\ldots r-1$. Each vacuum is characterised by a different effective coupling $\hat{\tau}=(r\tau_R+t)/s$. The exact superpotential in each of these vacua was calculated in [15]. For the SU(N) theory the result is,

$$\mathcal{W}_{\rm eff}=\frac{rN\mu M^2}{2\kappa^2\sin\beta}\,\frac{\theta_1'\left(\frac{r\beta}{2}|\hat{\tau}\right)}{\theta_1\left(\frac{r\beta}{2}|\hat{\tau}\right)}. \tag{7.4}$$

This result allows us to calculate the vacuum expectation values of chiral operators in these vacua. Two operators we will focus on are,

$$u=u_{(0,0,2)}=\frac{1}{N}\,{\rm Tr}_N\left[\Phi_3^2\right]\qquad w=u_{(1,1,1)}=\frac{1}{N}\,{\rm Tr}_N\left[\Phi_1\Phi_2\Phi_3\right]. \tag{7.5}$$

By standard Ward identities, we can calculate the VEVs of these operators by taking appropriate derivatives of the superpotential,

$$\langle u\rangle=\frac{1}{N}\frac{\partial\mathcal{W}_{\rm eff}}{\partial\mu}\qquad\langle w\rangle=-\frac{\exp\left(-\frac{i\beta}{2}\right)}{N}\left[\frac{\partial\mathcal{W}_{\rm eff}}{\partial\beta}+\frac{i}{2}\frac{\partial\mathcal{W}_{\rm eff}}{\partial\kappa}\right]. \tag{7.6}$$

In the massless the limit $\mu\to 0$, $M\to 0$, vacua of the massive theory go over to vacua of the massless β-deformed theory. Of course it is not necessarily true that all vacua of the massless theory can be obtained fom the very specific deformation we consider here. Fortunately, at least for the SU(N) theory, we will be able to construct a one-parameter family of vacua on each of the the new branches described above.[12]

We begin by translating the vacuum structure of the massless theory into predictions for the condensates of the operators u and w defined above. For generic values of β, the massless theory only has a Coulomb branch. In some of these vacua $\langle u\rangle\neq 0$, but $\langle w\rangle=0$ in all vacua. On the other hand, when $\beta=2\pi/n$ with $N=nm$, the massless theory develops a Higgs branch $\mathcal{H}_m$. On this branch generic vacua have $\langle w\rangle\neq 0$, but $\langle u\rangle$ vanishes in all cases. In this section, we have argued using S-duality, that the theory has an SL$(2,\mathbb{Z})$ multiplet of new branches all occuring at different values of β. As the operators u and w transform with definite modular weights, the generic vacua on *all* of these branches have $\langle w\rangle\neq 0$ and $\langle u\rangle=0$. In particular $\langle w\rangle$ can take an arbitrary complex value on each of these branches These branches also have other non-zero condensates, but to calculate these in the massive theory, we would need to extend the analysis of [15] by adding sources for the corresponding operators to the superpotential.

[12]Unfortunately, there are no vacua of the massive U(N) theory with similar limiting behaviour. Hence we cannot use the same arguments to demonstrate the existence of the new confining branches in this case. However the arguments given above, based on the S-duality of the massless β-deformed theory, apply equally to the U(N) and SU(N) theories.

We will now attempt to recover this vacuum structure of by taking the massless limit $\mu \to 0$, $M \to 0$. From (7.4) and (7.6), the VEVs of u and w are proportional to M^2 and μM^2 respectively and so naively they vanish in this limit. This conclusion is unavoidable, except for special points where the effective superpotential (7.4) diverges. The ratio of θ-functions appearing in (7.4) has a simple pole when the argument $r\beta/2$ vanishes. It is also quasi-periodic in this parameter with periods π and $\pi\hat{\tau}$, thus has one pole in each period parallelogram. In vacua with $t = 0$, we have $\hat{\tau} = r\tau_R/s$ where $N = rs$, and the simple poles are located at,

$$\beta = \frac{2\pi p}{r} + \frac{2\pi q \tau_R}{s} = \beta_{cr}(p, q|r, s) \tag{7.7}$$

for arbitrary integers p and q. Remarkably, these singular points coincide precisely with all the values of β where the arguments given above predict new branches. In particular, the Higgs branch $\mathcal{H}_m$ occurs at, $\beta = 2\pi/n = \beta_{cr}(1, 0|n, m)$. Restoring the integer $t = 0, 1 \ldots r-1$ we find a pole corresponding to a Higgs branch for $\beta = 2\pi l/N$ where l can be any integer. The dual magnetic branch $\tilde{\mathcal{H}}_m$ occurs at $\beta = 2\pi\tau_R/n = \beta_{cr}(0, 1|m, n)$. Near each critical point we have,

$$\mathcal{W}_{\rm eff} \sim \mu M^2 \frac{1}{(\beta - \beta_{cr})} . \tag{7.8}$$

Using (7.6) we find,

$$\langle u \rangle \sim M^2 \frac{1}{(\beta - \beta_{cr})} \qquad \langle w \rangle \sim \mu M^2 \frac{1}{(\beta - \beta_{cr})^2} \tag{7.9}$$

We now want to investigate the massless β-deformed theories at these special values of β. We begin by taking the limit $M \to 0$, $\beta \to \beta_{cr}$ holding μ fixed. Specifically we take the limit $\beta - \beta_{cr} = \epsilon \to 0$ with $M \sim \epsilon$. In this case we end up with vacua in which $\langle u \rangle = 0$ and $\langle w \rangle$ can take an arbitrary complex value. One can also check that the other condensates calculated in [15] remain finite in this limit. Finally we can take the limit $\mu \to 0$ holding $\langle w \rangle$ fixed. Thus we have confirmed the existence of a branch of vacua with $\langle w \rangle \neq 0$ for each critical value of β. Of course, our choice of limits now implies that these branches also exist in the theory with one non-zero mass (ie $M = 0$ and $\mu \neq 0$). This is not obvious but we check it explicitly in appendix B.

In order to understand the physics of the S-dual branch $\tilde{\mathcal{H}}_m$, it is useful to recall the discussion of the Higgs branch $\mathcal{H}_m$ given in section 3. We start from the undeformed $\mathcal{N} = 4$ theory on the one-dimensional submanifold of the Coulomb branch denoted $\mathcal{C}_1$ (see discussion above Eq (3.5)). As above, we set the vacuum angle θ to zero for simplicity. The spectrum of the theory at this point includes N^2 W-bosons with masses $M^{(ab)}$ as given in (3.5) and magnetic monopoles with masses $(4\pi/g^2)M^{(ab)}$, as well as the usual spectrum of dyons. At weak coupling $g^2 \ll 1/N$, introducing a real non-zero value for β leads to mass splittings in the W-boson multiplet with half of each multiplet getting β-dependent masses as given in (3.6). This results in the appearence of massless electrically charged states when $\beta = 2\pi/n$ where $N = mn$. At this point the theory develops a Higgs branch $\mathcal{H}_m$ on which these massless electric states condense breaking the U(N) gauge group to U(m). As discussed in section 3, neither the Higgs branch nor the Coulomb branch can be lifted by quantum effects. Similarly, the orbifold singularity where the two branches intersect cannot be resolved by quantum corrections and the two branches will continue to intersect for all values of the gauge coupling. Thus, although the masses of the W-bosons can be renormalized for $\beta \neq 0$, we expect that the singular point where they become massless will still exist in the full quantum theory

At the classical level, the masses of the magnetic monopoles are unaffected by a non-zero value of β. However, in the presence of non-zero β, these states, like the W-bosons, are no longer BPS and we would certainly expect their masses to recieve β-dependent quantum corrections. Under an S-duality transformation (7.3) the W-bosons and magnetic monopoles of the Coulomb branch theory are exchanged. The S-transformation also maps the theory discussed above with $\beta = 2\pi/n$ to an S-dual theory with $\beta = 8\pi^2 i/\tilde{g}_R^2 n$. Hence we can map the above statements about the weak-coupling spectrum of W-bosons in the first theory to a corresponding statement about the strong-coupling spectrum of magnetic monopoles in the second. In this way we can infer the existence of a point on the Coulomb branch of the dual theory with massless magnetically-charged states. By S-duality we also find a new branch of vacua $\tilde{\mathcal{H}}_m$ on which magnetically charged states condense breaking the dual magnetic gauge group down to a U(m) subgroup. Similar arguments indicate the existence of points where various dyonic states becomes massless and condense leading to an SL$(2, \mathbb{Z})$ multiplet of new branches in oblique confining phases.

Of course, all of this is very reminiscent of Seiberg-Witten theory [52]. Even though we have only $\mathcal{N} = 1$ supersymmetry in our case, the Coulomb branch action is characterized by an $N \times N$ matrix τ_{ij} of abelian couplings as in the $\mathcal{N} = 2$ case. As usual low-energy electric-magnetic duality can be encoded by identifying this matrix with the period matrix of a genus N Riemann surface. The relevant curve should be closely related to the one given in [53].

For the SU(N) theory, another heuristic way of understanding the phase in which each vacuum is realised is via the massive deformation discussed above. In particular, we obtained a point on each new branch as a limit of a massive vacuum. The massive vacua in question reduce to the known vacua of the $\mathcal{N} = 1^*$ theory for $\beta = 0$. The classical Higgs branch $\mathcal{H}_m$ is continuously related to the $\mathcal{N} = 1^*$ vacuum with effective coupling $\hat{\tau} = n\tau_R/m$. This vacuum is visable classically and has unbroken gauge symmetry SU(m). This matches the low-energy gauge symmetry on $\mathcal{H}_m$. In the $\mathcal{N} = 1^*$ theory the unbroken SU(m) becomes strongly coupled and is confined with a mass gap in the IR. On the Higgs branch $\mathcal{H}_m$, the unbroken gauge symmetry is realised in a non-abelian Coulomb phase. Thus, in the simultaneous massless and $\beta \to \beta_{cr} = 2\pi/n$ limit discussed above, the mass gap goes to zero and the unbroken SU(m) is deconfined.

The dual magnetic branch $\tilde{\mathcal{H}}_m$ corresponds to an $\mathcal{N} = 1^*$ vacuum with $\hat{\tau} = m\tau_R/n$. This is normally interpreted as a vacuum in which the gauge group is Higgsed from SU(N) down to SU(n) which is then confined in the IR [27]. In 't Hoofts classification of massive phases, this is indistinguishable from a phase in which SU(N) is confined down to SU(m) which is then Higgsed in the IR. In fact one can move smoothly between these phases by varying the relative mass scales of the electric and magnetic condensates in the vacuum. As we take the massless limit, the low-energy SU(m) symmetry is restored leaving a theory on $\tilde{\mathcal{H}}_m$ which has SU(N) confined down to SU(m) as expected.

On the Higgs branch $\mathcal{H}_m$ we found that chromomagnetic charges are confined by magnetic flux tubes of finite tension which become BPS-saturated in the large-n limit. The S-dual statement is that on the magnetic Higgs branch $\tilde{\mathcal{H}}_m$, we should find the confinement of chromoelectric charge by electric flux tubes of finite tension,

$$\tilde{T} = \frac{8\pi^2|\tilde{\alpha}_1||\tilde{\alpha}_2|}{\tilde{g}_R^2 n}. \tag{7.10}$$

Thus we interpret the dual Higgs branch $\tilde{\mathcal{H}}_m$ as one in which the original electric gauge group is confined down to a $\mathrm{U}(m)$ subgroup which remains in a non-abelian Coulomb phase. Moving onto the Coulomb branch of the effective theory yields interesting vacua in which the gauge group is further confined down to $\mathrm{U}(1)^m$.

The regime of ultra-strong coupling $\tilde{g}_R^2 \gg N$ on this confining branch is dual to the weakly-coupled theory on the Higgs branch studied above. In this regime, long chromoelectric strings emerge as BPS saturated solitons in a low-energy effective theory with sixteen supercharges. As in the Higgs phase, an obvious question is whether the theory has a regime where the strings become light. In the Higgs phase, the BPS formula (6.1) becomes exact in the $N \to \infty$ limit (with the gauge coupling g^2 held fixed). Thus, by S-duality, (7.10) gives the exact tension of the confining strings in the large-N limit. This again suggests the existence of interesting regimes of parameter space where the confining strings become light. One such regime is a 't Hooft large-N limit in which $N \to \infty$ with $\tilde{g}_R^2 n$ held fixed and large. This is very suggestive because of the standard lore about confining gauge theories in the 't Hooft limit: one expects an infinite tower of glueball states corresponding to the fluctuations of a light string. In section 9, we will make a concrete proposal for the dual string theory for the confining branch of the β-deformed theory.

8. String theory realisations

As in many other examples, string theory can provide important insights about the dynamics of the gauge theory described above. The first step is to realise the theory in question on the world-volume of a suitable D-brane configuration. The undeformed $\mathcal{N} = 4$ theory with gauge group $\mathrm{U}(N)$ can be realised on the world volume of N parallel D3 branes in type-IIB string theory. The four dimensional gauge coupling is given in terms of the string coupling, g_s, as $g^2 = 4\pi g_s$ and the θ-angle is related to the RR scalar field $C_{(0)}$ of the IIB theory so that we have,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + C_{(0)}. \tag{8.1}$$

With this identification the $\mathrm{SL}(2,\mathbb{Z})$ duality of the $\mathcal{N} = 4$ theory is mapped onto the S-duality of the IIB theory. To decouple the gauge theory, both from gravity and from the excited states of the open string we must take the limit $\alpha' \to 0$ holding g_s fixed. We must also rescale the spatial coordinates transverse to the brane keeping the masses of open strings stretched between the branes fixed in the $\alpha' \to 0$ limit. When the 't Hooft coupling $\lambda = g^2 N$ is small, the gauge theory is weakly coupled. If it is large, the gauge theory is strongly coupled and has a dual description in terms of IIB string theory on the near horizon geometry of the D3-branes, which is $AdS_5 \times S^5$ with N units of Ramond-Ramond five-form flux through S^5 [23].

Introducing the marginal β-deformation corresponds to turning on a particular background flux for the complexified three-form field strength of the theory,

$$G_{(3)} = F_{(3)} - \tau H_{(3)} . \tag{8.2}$$

The particular mode which must be turned on can be deduced from the standard dictionary between field theory operators and SUGRA fields provided by the AdS/CFT correspondence. Specifically the SUGRA fields are expanded in appropriate spherical harmonics of the SO(6) $\simeq$ SU(4) rotational symmetry of the directions transverse to the branes and these modes are identified with chiral operators transforming in the corresponding representations of the SU(4) R-symmetry of the $\mathcal{N} = 4$ theory. As discussed in section 2, the chiral operator coresponding to the β-deformation is part of the **45** of SU$(4)_R$. The string theory realisation of the β-deformation therefore involves turning on the corresponding spherical harmonic of $G_{(3)}$ with (complex) field strength proportional to β.

As we saw in section 3, turning on an arbitrarily small value of β can have a significant effect. Specifically when $N = nm$ with $n \gg 1$, choosing the value $\beta = 2\pi/n$ leads to the existence of a new Higgs branch $\mathcal{H}_m$ on which the U(N) gauge group is spontaneously broken to $H =$ U(m). As usual we can deduce the corresponding distribution of D3 branes by looking at the eigenvalues of the three complex scalar fields Φ_i. We consider the vacuum with scalar VEVs given by (3.16). The three matrices $U_{(n)}$ and $V_{(n)}$ and $W_{(n)}$ appearing in (3.16) have the same eigenvalues: the n distinct n'th roots of unity. We define complex coordinates on the six dimensions transverse to the branes as,

$$z_1 = x_1 + ix_4 = \rho_1 e^{i\psi_1} \qquad z_2 = x_2 + ix_5 = \rho_2 e^{i\psi_2} \qquad z_3 = x_3 + ix_6 = \rho_3 e^{i\psi_3} . \tag{8.3}$$

The simplest case is when only two of the three fields have VEVs. Hence we set $\alpha_3 = 0$ in (3.16). For large-n, the D3-branes are uniformly distributed on a square torus $T^2(r_1, r_2)$ in R^6 defined by $\rho_1 = r_1 = |\alpha_1|(2\pi\alpha')$, $\rho_2 = r_2 = |\alpha_2|(2\pi\alpha')$, $z_3 = 0$. Reintroducing non-zero α_3, has the effect of slanting the torus. In this more general case the resulting two-dimensional torus is defined by $\rho_i = r_i = |\alpha_i|(2\pi\alpha')$ for $i = 1, 2, 3$ with the extra condition,

$$\psi_1 + \psi_2 + \psi_3 = \delta_1 + \delta_2 + \delta_3 \tag{8.4}$$

where $\alpha_i = |\alpha_i| \exp(i\delta_i)$. In the following we will focus on the simplest case of a square torus, setting $\alpha_3 = 0$, unless otherwise stated.

The brane configuration described above carries N units of D3-brane charge. The fact that Φ_1 and Φ_2 do not commute implies that it also carries non-zero moments of D5-brane charge. This follows from the usual identification of the commutator of the scalar fields on a Dp-brane world-volume with D(p+2)-brane density. In fact, the VEVs of the scalar fields coincide precisely with those occuring in the standard M(atrix) theory construction of m toroidally wrapped D(p+2)-branes [49]. The corresponding M(atrix) lagrangian is that of the undeformed $\mathcal{N} = 4$ theory. In this context, the configuration is unstable at finite n, only becoming a true ground state in the $n \to \infty$ limit. In the present case, the presence of the β-deformation of the $\mathcal{N} = 4$ theory has the effect of rendering the configuration stable.

JHEP08(2004)043

At large-n, the brane configuration corresponding to our chosen vacuum therefore consists of m D5-branes wrapped on the torus $T^2(r_1, r_2)$. This is completely analogous to the appearance of spherically wrapped D5 branes in the string theory realisation of the $\mathcal{N} = 1^*$ theory [2]. In particular, the $\mathcal{N} = 1^*$ theory has a vacuum in which $\mathrm{U}(N)$ is broken to $\mathrm{U}(m)$, which corresponds to m D5 branes wrapped on a fuzzy sphere. Indeed, for the $\mathrm{SU}(N)$ theory, we argued above that this vacuum can actually be deformed[13] into a vacuum on the Higgs branch $\mathcal{H}_m$ by turning on the deformation parameter and taking the masses to zero. However there are also some important differences between the two cases. As mentioned in the introduction, toroidal compactification preserves supersymmetry while spherical compactification breaks most of it. Another important difference is that the $\mathcal{N} = 1^*$ vacua are isolated while in the massless β-deformed case we have a continuous degeneracy of vacua. Correpondingly, the fuzzy sphere's appearing in the $\mathcal{N} = 1^*$ case have fixed radii set by the mass parameters, while the fuzzy torii discussed above have variable radii set by the Higgs branch VEVs.

The worldvolume theory on the toroidally-wrapped D5-branes is an $\mathcal{N} = (1, 1)$ supersymmetric six-dimensional gauge theory with gauge group $\mathrm{U}(m)$. Thus we can already see the emergence of the six-dimensional effective theory derived in the previous sections. Compactification of this theory on a torus with periodic boundary conditions preserves all the supersymmetries. The presence of N units of D3 brane charge is realised on the D5-brane world-volume a constant non-zero background for the Neveu-Schwarz two-form B_{NS}.

$$\int_{T^2} B_{\mathrm{NS}} = 2\pi n(2\pi\alpha') \,. \tag{8.5}$$

We introduce flat coordinates χ_1 and χ_2 on the torus with $0 \leq \chi_1 \leq 2\pi r_1$ and $0 \leq \chi_2 \leq 2\pi r_2$. In this basis the metric on the torus is the flat one $g_{ij} = \delta_{ij}$. The two-form field can be written as,

$$B_{\mathrm{NS}} = \frac{n}{2\pi r_1 r_2}(2\pi\alpha')\, d\chi_1 \wedge d\chi_2 = \frac{n}{2\pi|\alpha_1||\alpha_2|}\,\frac{1}{(2\pi\alpha')}\, d\chi_1 \wedge d\chi_2 \tag{8.6}$$

and the corresponding 'flux' per unit area[14] of the torus is given by,

$$\mathcal{B} = \frac{1}{4\pi^2 r_1 r_2}\int_{T^2} B_{\mathrm{NS}} = \frac{n}{2\pi|\alpha_1||\alpha_2|}\,\frac{1}{(2\pi\alpha')} \,. \tag{8.7}$$

As usual, a non-zero value of B_{NS} on a D-brane world-volume induces non-commutativity. The low-energy theory on the D5 world-volume is the six-dimensional $\mathrm{U}(m)$ theory compactified down to four dimensions on a non-commutative torus. To identify the parameters of this non-commutative gauge theory, we follow the procedure advocated by Seiberg and Witten [50]. The metric g_{ij} introduced above is the metric on the torus as seen by closed strings. It is important to distinguish this metric from the effective metric G_{ij} which

[13]The appearance of extra massless modes as we approach the Higgs branch vacuum seems consistent with the fact that a sphere cannot be *continuously* deformed into a torus.

[14]The word flux appears here in inverted commas because B_{NS} is a two-form potential rather than a field-strength.

appears in the open string S-matrix elements. Similarly we must distinguish between the closed string coupling g_s and its open string counterpart G_s. The relation between the two sets of quantities is,

$$G_{ij} = -\left(B_{\rm NS} g^{-1} B_{\rm NS}\right)_{ij} \qquad G_s = g_s {\rm det}^{-\frac{1}{2}}\left(B_{\rm NS} g^{-1}\right) \tag{8.8}$$

The first relation in (8.8) implies that the radii of the torus as measured in the open string metric are,

$$R_1 = B_{\rm NS}^{12} r_2 = \frac{n}{2\pi|\alpha_1|} \qquad R_2 = -B_{\rm NS}^{21} r_1 = \frac{n}{2\pi|\alpha_2|}\,. \tag{8.9}$$

The second relation in (8.8) yields the six-dimensional coupling;

$$G_6^2 = G_s^2 = \frac{g^2 n}{|\alpha_1||\alpha_2|}\,. \tag{8.10}$$

If we choose 'open string' coordinates $\hat{x}_1$ and $\hat{x}_2$ on the torus with $0 \leq \hat{x}_1 \leq 2\pi R_1$ and $0 \leq \hat{x}_2 \leq 2\pi R_2$. These coordinates have non-vanishing commutator $[\hat{x}_i, \hat{x}_j] = i\vartheta$ with $\vartheta = B_{\rm NS}^{-1}$. This gives $\vartheta_{ij} = \vartheta\varepsilon_{ij}$ with,

$$\vartheta = \frac{n}{2\pi|\alpha_1||\alpha_2|}\,. \tag{8.11}$$

Thus the parameters of the low-energy theory on the D5 branes are in precise agreement with the results of our earlier field theory analysis.

This stringy derivation of the low-energy theory on the branes is valid when the closed string coupling is small: $g_s = g^2/4\pi \ll 1$. At large-N, this is a much weaker condition than the condition of small 't Hooft coupling, $g^2 N \ll 1$ required to justify the semiclassical field theory analysis of section 4. As in [23], it also includes a regime of strong 't Hooft coupling, where the worldvolume gauge theory has a dual description in terms of low-energy supergravity. The fact that the resulting parameters of the low-energy theory agree with the field theory results (5.1), (5.2), (5.3) is consistent with our claim that the Kaluza-Klein spectrum (4.4) becomes exact at large-N. The supergravity description of the brane configurations described in this section is studied in detail in [5].

In general, the gauge theory on D5 branes is still coupled both to gravity and to the excited states of the open string. In the present case a limit which decouples the worldvolume gauge theory from these extra degrees of freedom was identified in [54].[15] The limit is one in which $\alpha' \to 0$ and $\mathcal{B} \to \infty$ with g_s and $\alpha'\mathcal{B}$ held fixed. Here $\mathcal{B}$ is the background two-form 'flux' per unit area defined in (8.7) above. The resulting theory reduces to a six-dimensional U(m) gauge theory with non-commutativity in two space-like dimensions at low energies. At high-energies, it has a dual description in terms of IIB supergravity in the near-horizon geometry of the branes. For fixed N, the D5 brane decoupling limit is not the same as the decoupling limit which isolates the four-dimensional β-deformed theory on the original D3 branes. This does not present a puzzle because we know that the four-dimensional β-deformed theory should differ from its six-dimensional low-energy effective theory on length scales of order the lattice spacing. In section 9, we will

[15] See the $m = 1$ case of the discussion of D5 branes given in section 2.3 of this reference.

instead relate the decoupling limit for the D5 brane world-volume theory to a continuum limit in which the lattice spacing goes to zero.

The brane configuration described above also yields a nice way of understanding the confinement of magnetic charge discussed in section 5 [2]. We start from the $\mathcal{N}=4$ theory with gauge group $G = \mathrm{U}(N)$ as realised on the world-volume of N D3-branes. An external magnetic charge is represented as a semi-infinite D-string ending on the D3 brane. This corresponds to a single unit of non-abelian magnetic charge $k=1$ as measured by $k \in \pi_1[\mathrm{U}(1)] = \mathbb{Z}$. On the other hand, a single D-string parallel to the D3 world volume is absorbed onto the D3 worldvolume as one unit of magnetic flux through the transverse plane. As the $\mathcal{N}=4$ theory is in a non-abelian Coulomb phase the flux is not confined but spreads out as uniformly as possible

After introducing the β-deformation with $\beta = 2\pi/n$ and moving onto the Higgs branch $\mathcal{H}_m$, the N D3 branes are uniformly spread on the torus $T^2(r_1, r_2)$ and we also have m wrapped D5-branes. A D-string cannot end on a D5 brane but a D-string lying parallel to the D5 worldvolume can be absorbed as an instanton string in the six-dimensional world-volume theory on the D5s. The strings in question have a finite core-size set by the scale-size ρ of the instanton. A D-string which approaches the D5 world-volume along a perpendicular cannot end but it may bend through 90^0 and smoothly join onto a worldvolume string. If two D-strings of opposite orientation approach the D5 worldvolume, the lowest energy configuration is one in which they are smoothly joined in this way by a worldvolume instanton string. This corresponds to the confinement of magnetic charges by a linear potential whose slope is set by the tension of the instanton string.

For completeness we also consider the D-brane configurations which correspond to more general vacua in which the unbroken gauge group $\mathrm{U}(m)$ is further broken to its Cartan subalgebra. Specifically we consider the vacua characterized as in (3.17) by $m \times m$ diagonal matrices $\Lambda^{(i)}$ for $i = 1, 2, 3$. We choose the eigenvalues as $\lambda_s^{(i)} = \alpha_i \mu_s^{(i)}$ for $s = 1, 2, \ldots, m$ where $\mu_s^{(i)}$ are real numbers. As above we set $\alpha_3 = 0$ for simplicity. Hence, at large-N, the D3-branes are uniformly distributed on m concentric torii $T^2(r_s^{(1)}, r_s^{(2)})$ of radii $r_s^{(i)} = |\alpha_i| \mu_s^{(i)} (2\pi\alpha')$ for $i = 1, 2$. As above, we interpret these configurations as m D5 branes wrapped on the corresponding concentric torii each carrying n units of D3-brane charge. The light spectrum now includes W-bosons which correspond to open strings streched between torii with different radii. Extra massless states occur whenever pairs of torii coincide, corresponding to the restoration of a non-abelian subgroup of $\mathrm{U}(m)$.

So far we have restricted our attention to weak string coupling $g_s \ll 1$ but we would now like to consider what happens when the string coupling is increased. As discussed above, field theory considerations indicate that the Higgs branch cannot be lifted by quantum corrections. This suggests that the brane configuration described above remains stable as the string coupling is increased. As $g_s \to \infty$ we may use the S-duality of the IIB theory (which is equivalent to the field theory S-duality discussed above) to find a dual brane configuration in the weakly coupled string theory. Under S-duality the parameters of the IIB theory transform as,

$$g_s \to \tilde{g}_s = \frac{1}{g_s} \qquad \alpha' \to \tilde{\alpha}' = g_s \alpha' \, . \tag{8.12}$$

The m wrapped D5 branes are mapped onto m wrapped NS5-branes. Using the above transformation laws we can rewrite the radii of the torus in terms of the S-dual variables as $r_1 = |\tilde{\alpha}_1|(2\pi\tilde{\alpha}')$ and $r_2 = |\tilde{\alpha}_2|(2\pi\tilde{\alpha}')$. The N units of D3 brane charge are invariant under S-duality and can be realised on the NS5 worldvolume as a constant background for the Ramond sector two-form potential B_{RR}. More general Higgs vacua corresponding to D5 branes wrapped on concentric torii of different radii can also have S-duals with NS5 branes replacing the $D5$s. In this case, the spectrum includes massive 'magnetic W-bosons' corresponding to D-strings stretched between the NS5-branes of different radii. In each of these vacua, we now have a natural explanation for the confinement of electric charges which is dual to the corresponding discussion of magnetic charges given above. External electric charges correspond to fundamental strings. These objects cannot end on an NS 5-brane, but instead may form a bound-state with it which corresponds to an instanton string in the world-volume gauge theory. Thus the instanton strings are identified as chromoelectric flux tubes in agreement with our earlier field theory discussion

The theory also has complicated set of vacua in other phases. For example, we may act with the whole S-duality group on $\mathcal{H}_m$ to obtain branches of vacua in interesting oblique confining phases in which the unconfined low-energy gauge group is $\mathrm{U}(m)$. These vacua correspond to m (p,q) 5-branes wrapped on $T^2(r_1, r_2)$. For each co-prime values of p and q, the corresponding branch exists only for a particular value of β. Thus unlike the $\mathcal{N}=1^*$ case we never have any two of the branches coexisting in the same β-deformed theory. By moving onto the Coulomb branch of the low-energy theory we obtain a vacuum where the gauge group is Higgsed, confined or obliquely confined down to $\mathrm{U}(1)^m$. These branches are realised as m (p,q) 5-branes wrapped on concentric torii.

9. Towards a continuum limit

In section 4, we argued that the β-deformed theory on its Higgs branch $\mathcal{H}_m$ is equivalent to a six-dimensional lattice gauge theory. The parameters of the six dimensional theory are the lattice spacings, ε_1 and ε_2, the radii of compactification, R_1 and R_2, as well as the six-dimensional coupling, G_6, and non-commutativity parameter ϑ. These quantities are given in terms of the parameters of the four-dimensional theory in Equations (5.1), (5.2), (5.3) to which we now refer.

Given any lattice theory an obvious question is whether we can obtain an interacting continuum limit. In the present context, this means taking a limit in which $\varepsilon_i \to 0$, for $i = 1, 2$, with G_6 held fixed. In sections 9.1 and 9.2 we will discuss two such limits. In the final subsection we discuss continuum limits of the theory in the S-dual Higgs/confining phases where the unbroken/unconfined gauge group is $\mathrm{U}(1)^m$ A more detailed discussion of the results presented in this section is given in [1]

9.1 A matrix theory limit

To achieve a continuum limit with fixed non-zero six-dimensional coupling we can take $N \to \infty$ while holding $m = N/n$ and g^2 fixed. We therefore choose the scaling,

$$|\alpha_1| \sim \sqrt{n} \to \infty \qquad |\alpha_2| \sim \sqrt{n} \to \infty\,. \tag{9.1}$$

This means that G_6 and ϑ remain fixed. With these choices the radii $R_1 = n/2\pi|\alpha_1|$ and $R_2 = n/2\pi|\alpha_2|$ grow like $\sqrt{n}$ and the torus decompactifies. These scalings are similar to the standard Matrix theory construction of m D(p+2)-branes starting from N Dp-branes [37]. Note however that the four dimensional 't Hooft coupling diverges in this limit: $g^2 N \to \infty$. Thus we cannot justify the existence of the continuum limit purely on the basis of semiclassical field theory analysis. On the other hand, we have argued above that the whole spectrum of Kaluza-Klein states becomes BPS-saturated at large-N and the the mass formula (4.4) becomes exact. Thus the spectrum is consistent with the emergence of a continuum theory in six dimensions in the proposed continuum limit.

We will now consider what kind of continuum theory could arise in such a limit. Naively, we would obtain a continuum six-dimensional non-commutative gauge theory. Of course six-dimensional gauge theories are non-renormalizable, so this cannot be the whole story: any continuum limit must involve some new physics in the UV. In the commutative case, there is only one known non-gravitational theory which reduces to six-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory at low energy. This is the decoupled theory on coincident NS5 branes of the type-IIB theory which is known as Little String Theory [51]. A more general theory[16] which reduces at low energies to a continuum six-dimensional U(m) gauge theory with sixteen supercharges and non-commutativity in two spacelike dimensions was identified in [54]. For $g^2 \ll 1$, the UV behaviour of the theory is governed instead by IIB supergravity in the near-horizon geometry of the D5 branes with background $B_{\rm NS}$. The theory has one dimensionless parameter which controls the ratio of the non-commutative length scale and the six-dimensional gauge coupling. In a limit where this ratio goes to zero the theory reverts to the ordinary commutative Little String Theory.

It is not hard to motivate the appearance of the continuum six-dimensional theory described above in the present context. In fact we already encountered the theory in question in section 8, where it appeared as the Seiberg-Witten [50] decoupling limit of the world-volume gauge theory on the m D5 branes in the string theory construction of the Higgs branch $\mathcal{H}_m$ at large-N. The decoupling limit involved taking $\alpha' \to 0$ and $\mathcal{B} \to \infty$ simultaneously while holding g_s and $\alpha'^2\mathcal{B}$ fixed. As above, $\mathcal{B}$ is the background two-form 'flux' per unit area of $B_{\rm NS}$ defined in (8.7) above. In terms of field theory variables the quantities held fixed are g^2 and $n/|\alpha_1||\alpha_2|$ respectively. At a fixed value of N, the decoupling limit for β-deformed $\mathcal{N} = 4$ theory realised on N D3 branes is simply $\alpha' \to 0$. The fact that these limits are different reflects the fact that the β-deformed theory only agrees with the worldvolume theory on the D5 branes on lengthscales larger than the lattice spacing. However, if we combine the $\alpha' \to 0$ limit with the proposed continuum limit (9.1) in which the lattice spacing goes to zero, we have just the right scalings to decouple the six-dimensional gauge theory on the D5 branes. Thus we propose that, in the continuum limit described above, the β-deformed theory on its Higgs branch $\mathcal{H}_m$ is fully equivalent to the decoupled theory on the D5 branes.

[16] See the $m = 1$ D5 brane case discussed in section 2.3 of this reference as well as the S-dual discussion of NS 5 branes in section 3.2.

The resulting continuum theory in six dimensions has only one dimensionless parameter, the gauge coupling $g^2 = 4\pi g_s$. In the limit $g^2 \to \infty$, the length-scale of non-commutativity goes to zero and the theory becomes $\mathcal{N} = (1,1)$ Little String Theory. In string theory this limit is best understood by performing an S-duality transformation (8.12) which turns the m D5 branes into m NS5 branes with dual string coupling $\tilde{g}_s \to 0$ with $\tilde{\alpha}'$ held fixed. In field theory, the corresponding S-duality transformation yields a $\tilde{g}_R^2 \simeq \tilde{g}^2 \to 0$ limit of the theory on the magnetic Higgs branch $\tilde{\mathcal{H}}_m$ where the gauge group is confined down to $\mathrm{U}(m)$. The upshot is a proposal for a new way of deconstructing Little String Theory as a large-N limit of a confining gauge theory in four dimensions. The complete proposal is as follows;

- We consider the $\mathrm{U}(N)$ β-deformed theory with renormalized gauge coupling $\tilde{g}_R^2$, deformation parameter $\tilde{\beta} = 8\pi i/\tilde{g}_R^2 n$ (with $n|N$) and zero vacuum angle. This theory has a Higgs branch where the $\mathrm{U}(N)$ gauge group is confined down to a $\mathrm{U}(m)$ subgroup. We consider the vacuum on this branch specified as in (7.1), (7.2) by complex numbers $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ with $\tilde{\alpha}_3 = 0$.

- We take the limit $N \to \infty$ with m and $\tilde{g}_R^2$ held fixed, while scaling the VEVs as,

$$|\tilde{\alpha}_1| \sim \sqrt{n} \to \infty \qquad |\tilde{\alpha}_2| \sim \sqrt{n} \to \infty \tag{9.2}$$

  These scalings leave the BPS string tension $\tilde{T} = 8\pi^2|\tilde{\alpha}_1||\tilde{\alpha}_2|/\tilde{g}_R^2 n$ fixed.

- Finally we take the zero coupling limit $\tilde{g}_R^2 \simeq \tilde{g}^2 \to 0$ with the string tension $\tilde{T}$ held fixed. In this limit the world-volume non-commutativity disappears and the central $\mathrm{U}(1)$ of the gauge group decouples.

We propose that the resulting theory is the $\mathcal{N} = (1,1)$ Little String Theory with low-energy gauge group $\mathrm{SU}(m)$. This theory has a single dimensionful parameter: the tension of the little string. This is identified with the BPS string tension $\tilde{T}$ of the four-dimensional gauge theory.

9.2 Another continuum limit

We would also like to discuss a limit analogous to that considered in [17] which yields a commutative theory in six dimensions on a torus of fixed size. There is a unique way to take the lattice spacing to zero as $n \to \infty$ while holding R_1 and R_2 fixed, We must scale the VEVs as

$$|\alpha_1| \sim n \to \infty \qquad |\alpha_2| \sim n \to \infty\,. \tag{9.3}$$

This means that the non-commutativity parameter $\vartheta = n/2\pi|\alpha_1||\alpha_2| \sim 1/n$ and thus goes to zero as required. If we also want to hold $G_6^2 = |\alpha_1||\alpha_2|/g^2 n$ we are forced to take $g^2 \to \infty$ holding g^2/n fixed.

Like the limit discussed in the previous section, this limit can be reinterpreted using the S-duality of the β-deformed theory. It is equivalent to a simultaneous $N \to \infty$, $\tilde{g}_R^2 \to 0$ limit of a vacuum on the dual confining branch $\tilde{\mathcal{H}}_m$, taken holding the combination $\tilde{g}_R^2 N$ and the vacuum moduli $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ fixed. Hence the BPS string tension $\tilde{T}$ given in (7.10) also remains constant in the limit. In fact this is just a standard 't Hooft limit of the confining gauge theory in a fixed vacuum state on the branch $\tilde{\mathcal{H}}_m$.

In [1], we present evidence that the resulting continuum theory is again commutative $\mathcal{N} = (1,1)$ Little String Theory with string tension identified with the BPS string tension $\tilde{T}$. The new feature is that the Little String Theory is compactified to four dimensions on a torus of fixed radii R_1 and R_2.

9.3 Deconstructing the Coulomb branch

In the preceeding two subsections we have proposed limits where the dynamics of the β-deformed theory is described by Little String Theory. In particular, we focused on the large-N limit of the theory on confining branches where the unconfined subgroup of the gauge group is $\mathrm{U}(m)$. In these limits the central $\mathrm{U}(1)$ subgroup decouples and we obtain the Little String Theory on m coincident NS5 branes whose low-energy gauge group is $\mathrm{SU}(m)$. However, Little String Theory also has a Coulomb branch where the NS5-branes are separated breaking the low-energy gauge group to $\mathrm{U}(1)^{m-1}$. In this phase, the spectrum also contains 'W-bosons' corresponding to D-strings stretched between the fivebranes. In particular one can define a double-scaling limit [55], where the W-boson masses are kept fixed in the NS fivebrane decoupling limit $\tilde{g}_s \to 0$, with $\tilde{\alpha}'$ kept fixed as before. The resulting theory has a holographic dual which includes six flat directions and an internal space which is (roughly speaking) the product of a compact coset and the two-dimensional euclidean black-hole [57]. Importantly, as the starting point involved only Neveu-Schwarz branes, there are no background Ramond-Ramond fields. When the W-boson mass is much larger than the scale set by the string tension, the dilaton is everywhere small and it suffices to consider tree-level string theory in this background. The corresponding worldsheet theory is solvable and and the correlation functions can be calculated explicitly [55].

The two proposed continuum limits described above can easily be modified to give Little String Theory in its Coulomb phase. To accomplish this we simply need to move onto the Coulomb branch of the low-energy $\mathrm{U}(m)$ theory. This corresponds to a phase discussed above where the $\mathrm{U}(N)$ gauge group of the β-deformed theory is confined down to $\mathrm{U}(1)^m$. In this case the BPS spectrum of the low-energy theory will also include W-bosons. We can now take the same limits described in sections 9.1 and 9.2 with the ratio $x = M_W/\sqrt{\tilde{T}} \gg 1$ held fixed where M_W is the mass of the lightest magnetic W-boson and, as above, $\tilde{T}$ is the string tension. This is particularly clear in the string theory construction of section 8, where the corresponding configuration involves m NS5 branes wrapped on concentric torii of different radii. As in the set up of [55] the W-bosons correspond to D-strings stretched between NS5 branes. In the present case, their masses are controlled by the differences in radii of the toroidally-wrapped fivebranes. Each radius corresponds to a modulus of the configuration which can be varied independently.[17]

By taking the limit discussed in section 9.2 while keeping the W-boson mass fixed we arrive at a proposal for deconstructing double-scaled Little String Theory compactified on

[17]For example, one can adjust the VEVs so that the masses of W-bosons match those found in the vacuum configuration described in Eq (1.10) of the second reference in [55].

a torus of fixed size. The starting point is a 't Hooft large-N limit of a four-dimensional gauge theory in a phase where the $\mathrm{U}(N)$ gauge group is confined down to a product of $\mathrm{U}(1)$'s. As discussed in section 7, the theory on this branch has a semiclassical regime where we can explicitly exhibit confinement of electric charges due to the formation of BPS flux tubes of fixed core size. In this regime the scale set by the string tension is much larger than the W-boson mass. In the previous paragraph we have described another regime of the large-N theory on this branch where the strings become light compared to the W-bosons. The effective description of the new regime is given by double-scaled Little String Theory compactified on a torus. The correlation functions of double-scaled Little String Theory calculated in [55] are consistent with the existence of an infinite tower of stable, higher-spin states which behave like free particles when $x \gg 1$. This matches well with our expectations for the large-N glueball spectrum of a confining gauge theory containing only adjoint fields.[18] The β-deformed theory therefore seems to provide a new and surprisingly tractable version of the duality between large-N confining gauge fields and weakly interacting strings [56]. This is discussed further in [1].

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A. S-duality properties of the $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ theories

In the $\beta = 0$ case the difference between the theories with gauge groups $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ is trivial. When $\beta \neq 0$, the situation is much more complecated because the additional trace fields of the $\mathrm{U}(N)$ theory, denoted $a_i = \mathrm{Tr}_N\, \Phi_i/N$ for $i = 1, 2, 3$, are now coupled to the traceless $\mathrm{SU}(N)$ fields. The results of [15] take the simplest form for the $\mathrm{SU}(N)$ theory: the exact superpotential in the massive vacua was given in Eq (7.4) above. This expression (and the resulting condensates) transforms with a fixed modular weight under the modified $\mathrm{SL}(2, \mathbb{Z})$ transformations (2.13). Thus we have a direct generalization of the $\mathrm{SL}(2, \mathbb{Z})$ duality of the $\mathcal{N} = 4$ theory.

The corresponding situation in the $\mathrm{U}(N)$ theory is less straightforward. The $\mathrm{U}(N)$ effective superpotential itself does not have simple transformation properties but instead the combination,

$$\hat{\mathcal{W}}^{\mathrm{U}(N)}_{\mathrm{eff}} = \frac{\mathcal{W}^{\mathrm{U}(N)}_{\mathrm{eff}}}{1 - \frac{4\kappa^2 \sin^2\left(\frac{\beta}{2}\right)}{N\mu M^2}\mathcal{W}^{\mathrm{U}(N)}_{\mathrm{eff}}} \tag{A.1}$$

transforms in the same way as the $\mathrm{SU}(N)$ superpotential.[19] Operators with good modular properties can be defined by differentiating $\hat{\mathcal{W}}_{\mathrm{eff}}$ with respect to parameters. For example

[18] In fact, the similarity between the spectrum of double-scaled Little String Theory and that of a large-N gauge theory was noted in [55].

[19] In fact they are equal: $\hat{\mathcal{W}}^{\mathrm{U}(N)}_{\mathrm{eff}} = \mathcal{W}^{\mathrm{SU}(N)}_{\mathrm{eff}}$.

we can define an operator $\hat{O}$ with

$$\langle \hat{O} \rangle = \frac{\partial}{\partial \mu} \hat{\mathcal{W}}^{\mathrm{U}(N)}_{\mathrm{eff}} \tag{A.2}$$

which transforms with modular weight $(1,1)$, just as $\mathrm{Tr}_N\, \Phi_3^2$ does in the $\mathrm{SU}(N)$ theory. The difference between the two expressions is subleading in either the large-N or small-$|\beta|$ regimes which are the main concern of this paper. The modular behaviour of the two theories is the same in these regimes. This matches the results of the linearized analysis given in section 2 which applies equally to gauge groups $\mathrm{U}(N)$ and $\mathrm{SU}(N)$.

Away from the regime of large-N or small $|\beta|$, it seems that the operators transforming with good modular weights are different for the $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ theories. This complication arises because the trace field a_3 of the massive $\mathrm{U}(N)$ theory, gets an expectation value in each vacuum [15]. To be precise we have,

$$\langle a_3 \rangle = \frac{2\kappa \sin\left(\frac{\beta}{2}\right)}{NM\mu} \mathcal{W}^{\mathrm{U}(N)}_{\mathrm{eff}} \,. \tag{A.3}$$

Taking (A.1), (A.2), (A.3) together, we find,

$$\langle \hat{O} \rangle = \frac{\langle \mathrm{Tr}_N\, \Phi_3^2 \rangle^2}{\left(\langle \mathrm{Tr}_N\, \Phi_3^2 \rangle^2 - \frac{1}{N^2} \langle \mathrm{Tr}_N\, \Phi_3 \rangle^2\right)} \,. \tag{A.4}$$

Similar relations hold for other condensates. Thus, we see that the operators $\hat{O}$ and $\mathrm{Tr}_N\, \Phi_3^2$ will have very different transformation properties whenever $a_3 \neq 0$. On the other hand, if we can consistently set $a_3 = 0$ then the modular transformation properties of the $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ theories will be the same. In the massless β-deformed theory considered in this paper $\langle a_3 \rangle$ vanishes in all supersymmetric vacuum states. For this reason we believe that the modular properties of the *massless* β-deformed theories with gauge groups $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ are actually the same.

B. Classical branches of the massive theory

In this appendix we consider massive version of the β-deformed theory. We first consider introducing a non-zero mass μ for a single chiral multiplet. This corresponds to seting $M = 0$ in (2.10) which gives $\Delta\mathcal{W} = \mu\, \mathrm{Tr}_N\, \Phi_3^2$. Perhaps surprisingly, the classical Higgs branch $\mathcal{H}_1$ which occurs for $\beta = 2\pi/N$ is not lifted by this perturbation [8]. Setting,

$$\langle \Phi_1 \rangle = \alpha_1 U_{(N)} \qquad \langle \Phi_2 \rangle = \alpha_2 V_{(N)} + \gamma V^{\dagger}_{(N)} U^{\dagger\, 2}_{(N)} \qquad \langle \Phi_3 \rangle = \alpha_3 W_{(N)} \tag{B.1}$$

we find that the modified F-term vacuum equations are satisfied provided $\alpha_1 \gamma \kappa \sin(\pi/N) = \mu \alpha_3$. This configuration is not D-flat, but can be made so by a complex gauge transformation. Thus we see that a three-parameter family of vacua survives for $\mu \neq 0$. As the matrices appearing in (B.1) are traceless this statement is equally true for gauge groups $\mathrm{U}(N)$ and $\mathrm{SU}(N)$. This construction generalises easily to the other Higgs branches $\mathcal{H}_m$ for $m > 1$. Generic vacua have non-zero VEVs for the chiral operators x, y, z and w, but

JHEP08(2004)043

the equation (3.15) is modified. This is consistent because introducng μ explicitly breaks one of the U(1) R-symmetries of the β-deformed theory. In fact the modified equation describes a resolution of the orbifold, where one of the three fixed lines described above has been eliminated.

As in the massless theory, acting with the S-duality transformation (2.13) produces an $SL(2,\mathbb{Z})$ multiplet of new branches. As discussed in section 7, the fact that these vacua are still present in the theory with $\mu \neq 0$ is a non-trivial check on our discussion of the massless limit.

We now turn on both mass parameters M and μ. In this case the behaviour of the $U(N)$ and $SU(N)$ theories is different. The classical vacuum structure of both theories can be deduced from the results of [8] and [15]. In the $SU(N)$ theory, the branch (B.1) is lifted while a continuous degeneracy of vacua survives in the $U(N)$ theory. Moving away from $\beta = 2\pi/N$ both theories only have isolated classical vacua corresponding to deformed SU(2) representations. One can explicitly check that, in the $SU(N)$ theory, one of these isolated vacua approaches a point on the Higgs branch (B.1) as we take the $M \to 0$, $\beta \to 2\pi/N$ limit discussed in the text. In contrast, the corresponding vacuum of the $U(N)$ theory instead approaches the root of this branch, a point with $\langle w \rangle = 0$, in this limit.

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